APPROXIMATE COMPUTATIONS

SKINNER

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PRINCIPLES

OF

APPROXIMATE COMPUTATIONS

BY

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PRINCIPLES

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APPROXIMATE COMPUTATIONS.

INTRODUCTORY.

1. It is frequently necessary to perform arithmetical operations upon quantities whose numerical values cannot be taken with absolute accuracy, either because these values are incommensurable with unity or because they are the results of measurements made in Astronomy, Physics, Chemistry, Engineering, etc., with instruments capable of giving only a limited degree of precision. When the ordinary rules of arithmetic are applied to a computation involving such approximate quantities the operation is often very long and tedious, and when it is finished the operator may be in doubt as to the degree of accuracy of his final result; that is, as to the amount of error to which this result is liable from the errors of the quantities employed. The object of the following pages is to present some simple rules for conducting computations involving approximate quantities, in such a manner as to require the fewest figures and to show at once the degree of accuracy of the result. The methods are partly suggested, as also a few of the

examples, by the small French treatises of Martin, Babinet and Housel, etc., but many changes have been made in the statement and demonstration of the rules, and much not found in those treatises has been added.

2. In practice we commonly either know before beginning a computation what degree of accuracy will be sufficient in the result, or else we wish to compute a result with all the accuracy possible. Our computations accordingly come under one or the other of the following general problems:

First. A set of quantities being given, whose values may be taken as accurately as we please, to compute a result with any required accuracy.

Second. A set of quantities being given, each to a certain degree of approximation, to compute a result as accurately as the data allow.

It might happen that the data of a problem were only known with a limited approximation, while yet the result should be demanded with a certain required accuracy. If, however, it should be found by the methods which will be applied to the first of these general problems that the errors of the data must render it impossible to get the result as accurately as required, the example would then simply have to be made a case of the second general problem.

3. In computations involving approximate quantities it is important to adopt the following principle, viz.: to overstate, rather than understate, the error at any time committed. We shall thus be sure of not claiming for our results greater accuracy than they really possess.

By a *superior limit* of an approximate number we mean merely some number known to be larger than the true value of the number. By an *inferior limit* of an approximate number we mean merely some

number known to be smaller than the true value of the number. Thus if the true value of an approximate number were known to lie between 800 and 700, the former would be a superior and the latter an inferior limit of the approximate number.

4. By an *absolute error* we mean the amount which must be added to or subtracted from an approximate number to obtain the exact number. We are generally satisfied with stating that the error does not exceed a certain amount; that is, we assign a superior limit to the possible error. More often than otherwise this limit is taken at a unit or half a unit of the order of the last figure retained in the approximate number. For example, the length of the circumference of a circle whose diameter is 1, being $\pi = 3.14159265\ldots$, if we take 3.1415 as the approximate value we make an error less than 0.0001; if we take 3.1416 for the approximate value the error is less than $\frac{1}{2} \times 0.0001$. In all cases in which we know the figure which would follow those retained we may reduce the error to at most *half* a unit of the lowest order retained. For this purpose we increase by 1 the last figure retained whenever the following figure would equal or exceed 5, as illustrated in the value of $\pi = 3.1416$.

It may then often happen that the right hand figure of an approximate number will differ from the corresponding figure of the exact number. In what follows, however, when we speak of a number as having n exact figures we shall mean that the error is less than half a unit of the order of the nth figure from the left; thus the number 3.1416 will be regarded as the value of π with five exact figures.

By a *relative error* we mean the fraction expressing the ratio which the absolute error of an approximate

number bears to the exact number; for example, if instead of the number 403 we take an approximate value, say 400, the relative error of this approximate value will be $\frac{3}{403}$. If then N represent the exact value of a number, E the absolute error of an approximate value of the number, and R the relative error of the same approximate value, we have the following exact relation between the quantities N, E and R:

$$R = \frac{E}{N}; \qquad (1)$$

or,

$$E = RN. \qquad (2)$$

The relative error of a number is often of more importance than the absolute error; for an absolute error that would in some cases be of no consequence might in others be many times greater than could be tolerated. For example, we should not often expect a large building to be laid out so as to correspond to the specifications within less than a millimetre, but an error of a millimetre in the diameter of a portion of a thermometric tube would render the instrument worthless.

5. In practical examples, since it is often impossible to obtain the exact value of a quantity sought, the following problems occur with regard to absolute and relative errors:

PROBLEM I. A superior limit of the absolute error of a given result being known, it is required to assign a superior limit to the relative error of the same result.

RULE I. *Divide the known limit of the absolute error by an inferior limit of the result.*

Thus if R' be the required superior limit of the relative error of a result, the known limit of the absolute error being E, and $N - x$ being an inferior limit of the result, we shall have

$$R' = \frac{E}{N - x} \qquad (3)$$

The reason for this rule is obvious. If it were possible to obtain the exact value of the result, the relative error of an approximate result would be given by equation (1). But equation (3) will give a larger value than equation (1). Hence, (Art. 3), Rule 1 will give a superior limit of the relative error. To find R' neither N nor x need be exactly known, but we may take for $N - x$ any convenient number known to be less than the true value of N. For example, if we have the number 73.42 . . . as the result of a computation, and know that the absolute error is less than 0.10 we may evidently be sure that the relative error is less than $\frac{0.10}{70} = \frac{1}{700}$; that is, this is a superior limit of the relative error.

PROBLEM 2. A superior limit of the relative error of a given result being known, it is required to assign a superior limit to the absolute error of the same result.

RULE 2. *Multiply the known limit of the relative error by a superior limit of the result.*

Thus if E' be the required superior limit of the absolute error of a result, the known limit of the relative error being R, we have

$$E' = R(N + x). \qquad (4)$$

It is hardly necessary to explain the reason for the rule. Equation (2) would give the exact value of the absolute error of an approximate result if it were possible to obtain N exactly. But equation (4) will give a larger value than equation (2). Hence Rule 2 will give a superior limit of the absolute error. For example, if 47.25 . . . be the result of a computation, and it be known that the relative error is less than $\frac{1}{1000}$, we may evidently be sure that the absolute er-

ror is less than $\frac{50}{1000}$; that is, this is a superior limit of the absolute error.

PROBLEM 3. If a computation is to be made, so that the absolute error of the result shall not exceed a limit E, assigned in advance, it is required to determine how large a relative error we are at liberty to make in the work, or, in other words, to assign an *allowable* limit to the relative error of the result.

RULE 3. *Divide the assigned limit of the absolute error by a superior limit of the result.*

Thus if R'' be the required allowable limit of the relative error to be made, we shall have

$$R'' = \frac{E}{N+x}. \tag{5}$$

The reason for the rule is obvious. If it were possible to know the exact value of the result, we should evidently, from equation (1), be at liberty to make a relative error in the work equal to $\frac{E}{N}$. We shall therefore, *a fortiori*, not exceed the assigned limit of error if R'' does not exceed the smaller value $\frac{E}{N+x}$, given by Rule 3. For example, suppose we were to make any computation so that the absolute error of the result should not exceed 0.002, and suppose the result to be 7.8 . . ., we should evidently be at liberty to make a relative error in the computation equal to $\frac{0.002}{8}$; for from equation (2) the limit of the absolute error would then be $E = \frac{0.002}{8} \times 7.8 \ldots < 0.002$; that is, the actual absolute error would be less than the limit 0.002, assigned in advance.

PROBLEM 4. If a computation is to be made so that the relative error of the result shall not exceed a limit R, assigned in advance, it is required to determine an allowable limit to the absolute error of the result.

RULE 4. *Multiply the assigned limit of the relative error by an inferior limit of the result.*

Thus if E'' be the required allowable limit of the absolute error to be made, we shall have

$$E'' = R(N - x). \qquad (6)$$

The reason for the rule is simple. If the absolute error of the result is made not to exceed $E'' = R(N - x)$, the relative error cannot exceed $\frac{R(N-x)}{N}$, which is evidently less than R, the limit assigned in advance. For example, suppose we were to make a computation so that the relative error of the result should not exceed $\frac{1}{1000}$, and suppose the result to be 3.2 . . . ; we should evidently be at liberty to make an absolute error in the computation equal to $\frac{3}{1000} =$ 0.003; for from equation (1) the relative error would then be $\frac{0.003}{3.2\ldots} < \frac{1}{1000}$; that is, the actual relative error would be less than $\frac{1}{1000}$, the limit assigned in advance.

In the application of Rules 3 and 4 to practical examples we should evidently have to begin by determining, from inspection or rough calculation, some value that might safely be taken for $N + x$ or $N - x$, as the case might be.

6. If an approximate quantity be given, the absolute error of which is stated to be less than a certain number of units of the lowest order retained, it is evident that the relative error of the quantity will be independent of the position of the decimal point. For suppose we have the numbers 7.024 . . ., 702.4 . . . and 0.07024 . . ., stated to be correct within three of the lowest units retained in each, the limits of their relative errors will, by Rule 1, be respectively

$$\frac{0.003}{7}, \qquad \frac{0.3}{700}, \qquad \text{and} \qquad \frac{0.00003}{0.07},$$

and each of these limits is equal to $\frac{3}{7000}$. If therefore a number is given, whose absolute error does not exceed *one* unit of the nth order, we may evidently find a limit of the relative error by rejecting the decimal point from the number, and also all the figures following the nth, replacing with zeros all the other figures except one or two at the left, and dividing unity by the result regarded as a whole number. For example, suppose it be known that the absolute error of the number 3.1415126 does not exceed a unit of the order of the 5th figure from the left, we may say at once that the relative error is less than $\frac{1}{30000}$. Also, if a result be given with a certain number of exact figures, since its absolute error will then by definition not exceed *half* a unit of the lowest order retained, the limit of the relative error may evidently be found by disregarding the decimal point, doubling the number, replacing all the figures but one or two at the left by zeros, and dividing 1 by the result. Thus, if the number 76.24 be a result whose absolute error is less than half a unit of its lowest order, we may say at once that the relative error cannot exceed $\frac{1}{15000}$.

7. It may be shown that if the numerator of the relative error of a given result be unity, and its denominator contain n entire figures, then when the first significant figure of the result is less than that of the denominator of the relative error, the absolute error of the result is less than a unit of the nth order, counting to the right from the highest significant figure of the result, inclusive; and when the first figure of the result is equal to or greater than that of the denomi-

nator of the relative error, the absolute error will still be less than a unit of the $(n - 1)$th order of the result, counting as before. Suppose, for example, that the relative error of a result be known to be less than $\frac{1}{6000}$. Since the position of the decimal point in the result is immaterial in this connection, we may suppose, if we please, that the result has but one figure in its entire part. Then, if this figure be less than 6, the absolute error will evidently by Rule 2 be less than $\frac{6}{6000} = 0.001$; that is, the denominator of the relative error having 4 places the absolute error is less than a unit of the order of the 4th figure, counting from the highest significant figure of the result inclusive. But if the units figure of the number were equal to or greater than 6, we should still have the absolute error less than $\frac{10}{6000} < 0.01$; that is, the denominator of the relative error having 4 places, and the first significant figure of the result being greater than that of this denominator, the absolute error is less than a unit of the 3d place, counting from the highest significant figure of the result inclusive.

Since, from equation (1), the absolute error is proportional to the relative error, it also follows that when the first significant figure of a given result is less than that of half the denominator of its relative error, the numerator of this error being 1, and its denominator having n entire figures, the absolute error is less than *half* a unit of the nth order, counting as before, and when the first significant figure of the result is equal to or greater than that of half the denominator of the relative error, the absolute error is less than half a unit of the $(n - 1)$th order, counting in the same way. Thus, if the relative error of a result were equal to $\frac{1}{8000}$, and the result were equal say to 38.7449 . . ., the absolute error would be less than half a unit of the

4th place from the left; but if the result were equal say to 945.347 . . ., the absolute error would still be less than half a unit of the 3d place from the left.

8. It does not follow from the principles of the preceding article that if the absolute error of a result having more than n figures is known to be less than a unit or half a unit of the nth place we may reject all the figures that follow the nth and still have the absolute error of the result within the same limits. For, take the example given above, in which the relative error was supposed to be $\frac{1}{8000}$, and the result 38.7449 . . ., we should know that the absolute error was less than half a hundredth; but if we take simply 38.74 for the result, we evidently make a new error, nearly equal also to half a hundredth; and if the two errors happened to be in the same direction the error of the result 38.74 might evidently be double the error of the result 38.7449 . . . If then we wish to retain only a part of the figures of any approximate result already obtained, and to be able to state that it is still correct within a unit or half a unit of the lowest order retained, we have this caution to observe, viz: to consider whether the new error made by rejecting figures, plus the error already made in the computation, will not exceed a unit or half a unit of the lowest order retained.

It is evident that if we wish to retain only a limited number of figures in a result to be computed, we can make the final error less than a *unit* of the lowest order to be retained, by assigning, as the limit of error to be made in the computation, *half* a unit of this order; for the error to be made by rejecting the figures that would follow need never be greater than another half unit of the same order (Art. 4). But we cannot in a complicated computation always be sure of assigning

in advance a small enough limit to the error so as to be able to reject all the figures after the nth and still to say that the error is less than *half* a unit of the nth order; for we are sometimes liable to make an additional error nearly equal to a half unit of this order when we come to reject the figures that would follow.

9. PROBLEM 5. The numerator of an assigned limit * to the relative error of a required result being 1, and the denominator being entire, it is required to determine how many exact figures of the result must be computed.

RULE 5. *If the first significant figures of the required result be equal to or greater than those of half the denominator of the assigned relative error, compute as many figures in the result, counting from its highest significant one inclusive, as there are figures in the said half denominator; otherwise compute one additional figure.*

We may demonstrate the correctness of this rule as follows: Let the assigned limit of relative error of a result be, say, $\frac{1}{7000}$. Since the position of the decimal point in the result is not important in this connection, we may fix the limit of absolute error of the result at half a *simple unit*, and determine how large the result would have to be to bring the relative error within the assigned limit. It is evident from the definition of relative error that any number greater than 3500, exact within half a simple unit, would have a relative error less than $\frac{\frac{1}{2}}{3500} = \frac{1}{7000}$, but that any number less than 3500, the absolute error being half a simple unit, would have a relative error greater than $\frac{1}{7000}$; consequently if the first figures of the result were equal

* In speaking of a limit of error, either relative or absolute, we shall hereafter mean the superior limit numerically, unless otherwise stated. Also, when a result is required with n exact figures we mean that all the figures following the nth are to be rejected, the nth figure being adjusted, if necessary, so that the absolute error shall be less than half a unit of this order. Art. 4.

to or greater than 35, only four places would be needed, but if the first figures of the result were less than 35 another place would have to be taken. For example, if the square root of 2 were required, so that the relative error should be less than $\frac{1}{800}$, we should have to find four exact figures of the root, viz.: 1.414; but if the square root of 0.0019 were required with the same limit of relative error, we should need only three exact significant figures, viz.: 0.0436.

It is obvious that if the limit of absolute error of a required result were placed at a unit, instead of half a unit, of the lowest order to be retained, the number of figures necessary in the result could be determined by Rule 5, if the word *half*, wherever it occurs in the rule, were struck out. For example, if it be asked how many figures it is necessary to retain in $\sqrt{19}$, so that if it be simply known that the absolute error is less than a *unit* of the lowest order retained, the relative error shall be less than $\frac{1}{800}$, we see that three figures, viz.: 4.35, are not enough, but that four, viz.: 4.358, are sufficient.

10. It is worth noticing, that if exact quantities are added to or subtracted from approximate quantities the absolute error of the result will be the same as if only the approximate quantities were taken; and also that if an approximate quantity be multiplied or divided by an exact number the relative error will remain unchanged. Thus if π be taken equal to 3.1415, the sum of this plus or minus any exact number will have the same absolute error; and the product or quotient of $\pi = 3.1415$ by any exact number will have the same relative error; as is evident from Art. 6.

ADDITION.

11. PROBLEM 6. A set of quantities being proposed, whose values may be taken as accurately as we please, it is required to determine how far each must be computed, so that the absolute error of their sum shall not exceed half a unit of the nth order.

RULE 6. *If there are not more than ten quantities to be added, compute each so that its absolute error shall not exceed half a unit of the order next below the* n*th. If there are more than ten quantities to be added, indicate by the plus or minus sign the direction of the error of each quantity when computed as just stated, or else compute in each still another figure.*

The rule needs but little explanation. If there are not more than ten errors in either direction, each not greater than, say, 0.005, or half a hundredth, the sum of all these errors will not exceed 0.05, or half a tenth.

If we wish to indicate the direction of the error of an approximate quantity, we place a plus sign after it when its error would have to be added to give the true result, and a minus sign in the opposite case.

EXAMPLE 1. Find the sum of the square roots of the numbers 2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 17, and 18, so that the absolute error of the result shall be less than 0.0005.

Taking each with four exact decimals, we have

$$
\begin{array}{rl}
1.4142 & + \\
1.7321 & - \\
2.2361 & - \\
2.4495 & - \\
2.6458 & - \\
2.8284 & + \\
3.1623 & - \\
3.3166 & + \\
3.4641 & + \\
3.6056 & - \\
3.7417 & - \\
3.8730 & - \\
4.1231 & + \\
4.2426 & + \\
\hline
42.8351 &
\end{array}
$$

There being in the above example not more than eight errors in one direction, each less than 0.00005, the error of the sum cannot exceed 0.0004, which is less than the assigned limit 0.0005. Observe, also, that in this particular example, by finding this smaller limit, viz.: 0.0004, since the last figure of the sum happens to be 0.0001, we are able to reject this and still to say that 42.835 is correct within 0.0005, or that this is the answer with five exact figures. But if we take the sum of the same numbers, omitting the last one, we find it to be 38.5925. We are therefore evidently unable, without knowing the direction of the error, to reject the last figure of this result, and still have the answer contain five exact figures.

In general when we wish to retain all the exact figures of an approximate result, and no more, we must find between what limits the true result lies, and then retain only those figures which would be kept for numbers at either limit. For example, in the case just cited, where the sum of thirteen numbers is

38.5925, eight of them are taken too large and five too small; hence the true result must lie between 38.5921 and 38.5928. We can therefore take 38.59 as the result with four exact figures, but by our definition neither 38.592 nor 38.593 as the result with five exact figures, for the computation does not show which of these results is the more accurate.

EXAMPLE 2. Compute the series

$$\frac{2}{3}+\frac{2}{3\cdot 3^3}+\frac{2}{5\cdot 3^5}+\frac{2}{7\cdot 3^7}+\frac{2}{9\cdot 3^9}+\ldots,$$

retaining only eight decimals in the result, and make the error of the result less than a unit of the eighth decimal place.

Observing the remark near the end of Art. 8, and taking nine terms, each exact to the ninth decimal place, we have,

$$\begin{array}{rl}
0.666666667 & - \\
24691358 & + \\
1646091 & - \\
130642 & - \\
11290 & + \\
1026 & + \\
96 & + \\
9 & + \\
1 & - \\
\hline
0.693147180 &
\end{array}$$

Answer, 0.69314718.

EXAMPLE 3. Compute the expression $\sqrt{44}+\sqrt{0.08}+\sqrt{0.06}+1.7435013\ldots$, with three exact decimals in the result.

Taking four exact decimals in each term the sum will be found to be 8.9044. But the possible error of this result is greater than 0.0001. Hence the last decimal cannot be rejected without determining still

another one. (Art. 8, at end). Let the next decimal be determined.

12. It is clear that if any other limit of absolute error be assigned for a sum than that stated in Problem 6, the allowable limit of absolute error for each of the quantities to be added may be found by dividing the assigned limit of absolute error of the sum by the number of quantities to be added.

EXAMPLE 4. Compute the expression $352.7856\ldots + \sqrt{2} + \sqrt{7} + 25.00082\ldots + 0.000074\ldots$, with a relative error in the result not to exceed $\frac{1}{50000}$.

We see that the sum will be greater than 375. Hence, by Rule 4, we are allowed to make an absolute error in the sum equal to $\frac{375}{50000}$, and as there are five numbers to be added we are evidently at liberty to make an absolute error in each of them equal to $\frac{1}{5} \times \frac{375}{50000} = \frac{75}{50000}$, which is greater than 0.001. Hence, if the thousandths place in each number be found within a unit of that order, the result will satisfy the conditions. Taking the first four numbers with that approximation, and neglecting the fifth, we have

$$\begin{array}{r} 352.785 \\ 1.414 \\ 2.645 \\ 25.000 \\ \hline 381.844 \end{array}$$

with an absolute error less than 0.005, and a relative error, therefore, less than $\frac{5}{300000} < \frac{1}{50000}$.

EXAMPLE 5. Calculate the sum of the reciprocals of the numbers 3, 7, 9, 11, 13, 14, 17, 18, 19, 21, 22, 23, 26, 27, 28, and 29, so that the relative error of the result shall be less than $\frac{1}{20000}$.

13. It is evident that if it be required to find the sum of several quantities which can each be obtained with only a limited degree of approximation, if we do not know the direction of the errors of any of them, we must take as the limit of the absolute error of the sum, the sum of the possible absolute errors of the quantities. If the absolute error of one of the quantities greatly exceeds those of the others we should commonly not take the trouble to find the exact sum of the errors; but we may take some approximate value of it that we can see would exceed it. For example, in adding the numbers 76.3, 18.71, and 528.345, supposed approximate within half a unit of the lowest order in each, we should be satisfied with saying that the sum could be found exact within 0.06.

EXAMPLE 6. Add the following numbers, supposed approximate within 2 units of the lowest order in each, and assign a limit to the absolute and relative error of the sum: 35.278, 26.435, 18.7346, 21.0064, 3.2178, 0.2142, 0.00125.

SUBTRACTION.

14. PROBLEM 7. Two quantities being proposed whose values may be taken as accurately as we please, it is required to determine how far each must be computed so that the absolute error of their difference shall not exceed an assigned limit.

RULE 7. *Compute each quantity so that its absolute error shall not exceed half the limit of error assigned for the difference.*

The rule needs no demonstration. And it is also evident that if the errors of the two quantities were in the same direction, these errors would not need to be made smaller than the limit assigned for the error of the difference. Thus, if a difference of two numbers be required within a unit of a given order, it will be sufficient to compute each to that order of units whenever it can be seen that their errors will be in the same direction.

EXAMPLE 7. Compute the expression $\sqrt{7} - \sqrt{5}$ so that the absolute error of the result shall not exceed 0.001.

The answer may be either 0.409 or 0.410.

EXAMPLE 8. Compute the expression $\sqrt{95} - \sqrt{5}$, with a relative error less than $\frac{1}{10000}$.

By calculating the first two figures of each root we find the difference will be greater than 7. Hence we are at liberty to make an absolute error in the result equal to 0.0007. But not knowing beforehand the

amount of the errors that would result by taking both numbers too small or both too large, with three decimals each, nor the direction of the errors if each were to be taken to the *nearest* 3d decimal, we have to determine in each the 4th decimal, giving

$$\begin{array}{r} 9.7468\ - \\ \underline{2.2361\ -} \\ 7.5107 \end{array}$$

Since the errors of the two quantities happen now to be in the same direction, the absolute error of the result cannot exceed 0.00005. Then the true difference lies between 7.5106 and 7.5108. Either of these limits would give for the answer with four exact figures, 7.511, the absolute error being then less than half a thousandth, and the relative error, therefore, (Art. 6) less than $\frac{1}{15000} < \frac{1}{10000}$ as required.

EXAMPLE 9. Compute the expression $\sqrt[3]{3} - \sqrt{2}$, with a relative error less than $\frac{1}{1000}$.

Answer, 0.02804.

15. It hardly needs to be pointed out that if two numbers can be obtained, each with only a limited degree of approximation, the limit of the absolute error of the difference of the numbers will have to be taken equal to the sum of the limits of the errors of the numbers, unless it is known that their errors are in the same direction; in this case the larger of the two limits of error may be taken as the limit of error of the difference. For example, if the numbers 3.725 and 1.834 are each approximate within half a thousandth, but the direction of the errors not known, there will be an uncertainty of a thousandth in the difference, 1.891. If, however, we have the numbers 3.725 and 1.8342 each known to be *too small* by not more than half a unit of its lowest order, the error of the difference 1.8908 cannot exceed half a thousandth.

MULTIPLICATION.

16. PROBLEM 8. Two factors being proposed, whose values may be taken as accurately as we please, it is required to form their product so that its absolute error shall not exceed a unit of the nth order of decimals.

RULE 8. *Take either factor for the multiplier, and write it with its figures in reverse order under the multiplicand, and in such a position that the original simple units figure of the multiplier shall come under the* (n + 1)*th decimal of the multiplicand. Begin each partial product with the product of the multiplying figure into the figure of the multiplicand directly over it, rejecting the figures of the multiplicand to the right of this, but correcting this product, if necessary by adding to it the number of units of the same ord[e]r nearest to what would be added if the rest of the multiplicand were used, and place the partial products with their right hand figures in a vertical line. The sum of the partial products will have* n + 1 *decimals, the last of which must generally be regarded as doubtful.*

EXAMPLE 10. Compute the product

$$763.05403698956 \times 25.4463057845$$

with an absolute error less than 0.0001.

Operation:

```
  763.0 540 3698956
548750 3 644.52
-----------------
   1526 1 080 74
    381 5 270 18
     30 5 221 61
      3 0 522 16
        4 578 32
          228 92
            3 82
              53
               6
-----------------
   1941 6.906 34
```

This method of abridged multiplication is ascribed to Oughtred. The explanation of it is very simple. In the above example the original units figure 5 of the multiplier stands under the fifth decimal figure of the multiplicand; hence, considering first the second partial product its right hand figure will be of the fifth order of decimals. But the original tens figure of the multiplier stands under the sixth decimal of the multiplicand, and the first *decimal* of the multiplier under the fourth decimal of the multiplicand, so that the partial products made with these figures of the multiplier will also begin at the right with the fifth decimal place; and so for all the partial products, since the position value of the figures of the reversed multiplier diminishes towards the left, in the same ratio as that of the figures of the multiplicand increases. The position of the decimal point in the result is easy to fix. It may in any case be determined by the last sentence of Rule 8. If the multiplier in any example has no entire figures, supply the place of simple units with a zero.

Let us consider the possible error of a result found by Rule 8. If the rule with regard to correcting the partial products to allow for the part of the multiplicand each time rejected is carefully followed, it is evident that the error of each partial product cannot exceed half a unit of its lowest order; and since the right hand figure of each partial product is of the same order as that of the final result, the error of this result, whenever the number of partial errors does not exceed twenty, cannot be greater than ten units of the lowest order obtained; that is, it will not exceed *one* unit of the next higher order, which is the limit assigned in the statement of the problem.

In the example worked above, the last partial product comes from using the figure 8 of the multiplier; and if the next figure of the multiplier at the left had been large enough to give more than half a unit of the lowest order in the result, we should have put in another partial product equal to one unit of that order. But since the number of partial products does not exceed ten, the total error of the result cannot exceed five units of its lowest order, and we may therefore reject the last figure 4 of the result, and the answer will still be correct within 0.0001, the assigned limit of error.

17. It is easy in practice to indicate the direction of the errors of the partial products by the plus and minus signs, and thus, when the work is done, to often determine a much smaller limit of error than that assigned in advance.

EXAMPLE 10*a*. Make the product of the same factors as in Example 10, with the same limit of error, but taking the former multiplier for the multiplicand.

$$
\begin{array}{rl}
25.4463\,057845 & \\
\underline{6598963\,0450.367} & \\
178\,1241\,405 & - \\
15\,2677\,835 & - \\
7633\,892 & - \\
127\,232 & - \\
10\,179 & - \\
76 & + \\
15 & + \\
\underline{2} & + \\
19416.90\,636 &
\end{array}
$$

By thus marking the direction of the errors we observe that five of the partial products are too large, and three are too small; hence the error of the result cannot be more than $2\frac{1}{2}$ units of the lowest order obtained; which is only a quarter of the limit of error assigned in advance. But without determining a still smaller limit of error, we could not decide from either this operation or that of example 9, or both of them, whether, if we wished to reject the last decimal of the result, the one before it should be left a 3 or changed to 4.

18. EXAMPLE 11. Multiply 0.995 .... by 9.95 .... so that the absolute error of the result shall not exceed 0.01.

By the rule for arrangement we have

$$
\begin{array}{r}
0.99\,5\ldots\ldots \\
\underline{\ldots\ldots59.9} \\
8\,95\,5 \\
89\,6 \\
\underline{5\,0} \\
9.90\,1
\end{array}
$$

But let us examine the possible error of this result,

supposing the 5 in each factor to be liable to an error of half a unit of its own order. The first partial product would evidently be liable to an error of $4\frac{1}{2}$ units of the order of the last place in the product; the second partial product would be liable to an error of nearly a unit of the same order, since, besides the error of the multiplicand, we make an additional error in rejecting a figure of this partial product; and, making a similar observation, the last partial product might be wrong by about 5 units of the same order. It is therefore possible that we have exceeded the assigned limit of error. In fact, if the true values of the numbers were 0.9945 and 9.945, their exact product would be 9.8903025, a result differing from the approximate one found above, by 0.0106975, or a trifle more than the assigned limit of error.

It is thus seen that in the application of Rule 8 it will not in all cases be enough to know within a half unit of its own order the figure of the multiplicand standing over the right hand figure of the reversed multiplier, and that of the multiplier standing under the highest figure of the multiplicand. But a single additional figure in each factor will be amply sufficient. For if the multiplicand has one exact figure to the right of the multiplier, and the multiplier one to the left of the multiplicand, we may always reduce the errors of the first and last partial products each to a single unit of the lowest order obtained in the product; and since the assigned limit of error is 10 of these lowest units there is still room for 16 additional partial products with errors all in the same direction, each less than half of one of these units.

19. PROBLEM 9. The product of two factors being required within a unit of the nth order of decimals, it is required to state a rule for determining the number of decimal places to be computed in each factor.

RULE 9. *Either factor being regarded as the multiplicand, compute one more decimal place in the other factor than the whole number of significant figures in the multiplicand above the* n*th order of decimals inclusive.**

This rule follows from the arrangement of the factors by Rule 8. According to that Rule the units figure of the multiplier will stand under the $(n + 1)$th decimal of the multiplicand. Rule 9 will then give us one figure of the multiplier to the left of the highest significant figure of the multiplicand, as required by the last section. Thus if it were required to multiply 0.007425625 by 99.284376 with an absolute error less than a unit of the 4th decimal place, the arrangement by Rule 8 would be

$$\begin{array}{r} 0.0074\ 25625 \\ \underline{67\ 3482.99} \end{array}$$

from which it is evident that the three left hand figures of the multiplier will not be used in forming the product. Striking them off, the number of *decimals* remaining in the multiplier, viz.: three, will exceed by one the number of significant figures in the multiplicand above its 4th decimal place inclusive, viz.: two. The number of necessary figures in either factor will not be altered by making the former multiplier the multiplicand. For, making this change we should have

$$\begin{array}{r} 99.2843\ 76 \\ \underline{52652\ 4700.0} \end{array}$$

In this arrangement the two left hand figures of the

* Whenever we speak of significant figures in this way we of course include any zeros that come in below the highest significant figure. Thus the number 0.0010702 would be regarded as having five significant figures.

new multiplier may be struck off, and the number of decimals left in it, viz.: seven, will again exceed by one the number of significant figures in the new multiplicand above its 4th decimal place inclusive, viz.: six.

EXAMPLE 12. Determine by Rule 9 how many decimal places would have to be calculated in $\sqrt{9057}$ and $\sqrt{0.0093}$ so that their product by Rule 8 should not be wrong by more than a unit of the 5th decimal place.

Regarding $\sqrt{9057}$ as the multiplicand, it will have seven figures above the 5th decimal place inclusive. Hence the number of decimal places to calculate in $\sqrt{0.0093}$ will be eight. Regarding $\sqrt{0.0093}$ as the multiplicand it will have four significant figures above the 5th decimal place inclusive. Hence the number of decimal places to calculate in $\sqrt{9057}$ will be five. Let the figures be obtained and the product formed by Rule 8.

20. Rule 9 is framed to cover safely all cases. But in a great majority of examples, viz.: where the sum of the highest significant figures of the two factors, plus one-half the number of partial products, does not exceed 18, we may do with one less exact figure; and if the sum of the highest significant figures of the two factors, plus one-half the number of partial products, is less than 8, we shall even then generally get the result with an error less than *half* a unit of the nth order. It is easy in any special example to recognize the possible error committed.

EXAMPLE 13. Compute the expression $\sqrt{0.0003} \times \sqrt{1000}$ with a relative error less than $\frac{1}{10000}$.

The square root of 0.0003 is more than 0.017, and that of 1000 is more than 30. Hence the product will be more than 0.5. By Rule 4 we are at liberty then to make an absolute error in the result equal to $\frac{0.5}{10000}$ = 0.00005, that is, half a unit of the 4th decimal place. The sum of the highest significant figures of the factors, 1 and 3, is so small that we shall probably be safe if we work as if the allowable error were a whole unit of the 4th decimal place, and take also one less decimal in each factor than Rule 9 would require. (The reason why the error of a result is less when the sum of the highest significant figures of the factors is small, is evidently that the sum of the errors of the first and last partial products will then also be small). Since $\sqrt{0.0003}$ will have three significant figures above the 4th decimal place inclusive, we must then know three exact decimals in $\sqrt{1000}$; and we may determine the requisite number of decimals in $\sqrt{0.0003}$ in a similar way, or by simply considering that we want the same number of significant figures in it as in $\sqrt{1000}$, which, from what has just been found, must be five. We then want the 6th decimal place in $\sqrt{0.0003}$.

The multiplication will be as follows:

$$
\begin{array}{rl}
31.623 & - \\
-\ \ 12\,3710.0 & \\
\hline
31\,623 & - \\
22\,136 & \mp \\
949 & - \\
63 & + \\
3 & \mp \\
\hline
0.54\,774 &
\end{array}
$$

In taking each factor in this example to the nearest unit of the lowest order retained, each is too large. If we suppose the multiplicand too large by half a unit of its lowest order, the first partial product would also be too large by half a unit of its lowest order, since the multiplying figure is 1. And if we suppose the multiplier too large by half a unit of its lowest order, the last partial product would be too large by about $1\frac{1}{2}$ units of the same order as before, since the highest figure of the multiplicand is 3. The errors of the second and third partial products cannot increase the error of the result to more than 3 units of its lowest order; hence the true result cannot be less than 0.54771. On the other hand, if the errors of the factors are as near zero as possible the sum of the three partial products which could then be too small will be less than a unit of the lowest order of the result. Hence the true result cannot be greater than 0.54775. We may then, if we please, reject the last decimal obtained in the result, and the answer, 0.5477 will have four exact figures, and a relative error less than $\frac{5}{54000} < \frac{1}{10000}$, as required.

EXAMPLE 14. Compute the expression $100\,\pi\,\sqrt{2}$, with a relative error in the result not exceeding $\frac{1}{10000}$.

The product will exceed 400. Then the allowable absolute error is $\frac{400}{10000} = 0.04$. If we work then as if the limit of error were *one* unit of the second decimal place, instead of four, we may safely determine the requisite number of decimals by taking one less than Rule 9 would give. Regarding $100\,\pi$ as a single factor, it will have five figures including the 2d decimal place, and we therefore need five decimals in $\sqrt{2}$. Computing these figures, and taking the value of $100\,\pi$ to correspond we have

$$
\begin{array}{rl}
 & 314.159\ + \\
+ & \underline{124\ 14.1} \\
 & 314\ 159\ + \\
 & 125\ 664\ - \\
 & 3\ 142\ - \\
 & 1\ 257\ - \\
 & 63\ - \\
 & \underline{3}\ + \\
 & 444.288
\end{array}
$$

If the 6th decimal figure of the multiplier be supposed equal to 5, there would be another partial product not exceeding 2 of the lowest units of the product. As for the partial products written, only two are too small. Hence the result cannot be too small by more than 3 of its lowest units. And it cannot be too large by more than 2 of these units. Therefore the true value of the result lies between 444.291 and 444.286. Hence the result to the nearest hundredth is 444.29, with a relative error less than $\frac{4}{440000} < \frac{1}{100000}$.

21. It is easy to see that if the highest significant figure of either factor is in the *n*th decimal place, then by Rule 9 there would be 2 decimals to calculate in the other factor. If the highest significant figure of one factor is p places below the nth there will then be $2 - p$ decimals to be calculated in the other factor; and the extension of Rule 9 to cases in which the assigned limit of error is a unit of higher order than decimals is therefore easy.

EXAMPLE 15. How many figures must be taken in the factors 9843.768 and 947.84321, so that the error of the product shall be less than a million?

The highest figure of the first factor stands in the third place below millions, hence the required num-

ber of decimals in the other factor is $2 - 3 = -1$; that is, we may *neglect the units figure.* The highest figure of the second factor is in the fourth place below millions, hence the number of decimals required in the first factor is $2 - 4 = -2$; that is, we may *neglect the tens.* We have then

$$\begin{array}{r} 98 \\ .059 \\ \hline 88 \\ 5 \\ \hline 93 \end{array}$$

Since the original units figure of the multiplier would thus come under the place of hundred thousands in the multiplicand, the right hand figure of the product obtained is hundred thousands; hence the product is 9300000, with an error less than a million.

EXAMPLE 16. How many figures must by this extension of Rule 9 be computed in each of the factors, $\sqrt{98734216}$ and $\sqrt{0.0093}$, so that the error of the product by Rule 8 shall be less than a unit of tens place.

The simple units in the value of the first factor may be disregarded, and we want four decimal places in the value of the second factor. Let the factors be found with this approximation, and their product formed by Rule 8.

EXAMPLE 17. How many figures must be computed in $\sqrt{758425}$ and $\sqrt{0.00009}$, so that the error of their product by Rule 8 shall be less than 100?

In Article 52 will be found a general rule for determining the necessary approximation of each factor where an expression contains more than two factors, either as multipliers or divisors or both.

22. PROBLEM 10. Two factors being given, each to a certain degree of approximation, it is required to assign a superior limit to the absolute error of their product.

RULE 10. *Multiply a superior limit of each factor by the absolute error of the other factor. The sum of the two products thus obtained may be taken for the limit of the absolute error of the product of the factors.*

Demonstration. Suppose there be given the two approximate factors, a' and b', whose absolute errors are α and β, we are to determine a limit of the error of the product, $a'b'$. In the most unfavorable case the errors will be in the same direction; if then a and b are the true values of the factors, suppose $a' = a + \alpha$, $b' = b + \beta$. Multiplying these two equations, member by member, the product will be

$$a'b' = ab + a\beta + b\alpha + \alpha\beta,$$

and the absolute error of the product will be

$$a'b' - ab = a\beta + b\alpha + \alpha\beta. \qquad (7)$$

Now since α and β are usually very small compared with a and b, the product $\alpha\beta$ will be very small compared with $a\beta$ and $b\alpha$, so that we have for the absolute error of the product very nearly

$$a'b' - ab = a\beta + b\alpha. \qquad (8)$$

In finding the limit of error by Rule 10 we shall evidently get a larger limit than would be given by the right hand member of equation (8), since instead of a and b we take by the rule quantities known to be larger than them. Hence the rule is a safe one, notwithstanding the neglect of the very small quantity $\alpha\beta$.

EXAMPLE 18. Determine the limit of absolute error in the product of the factor 784.2817, supposed to have an absolute error not exceeding 0.0004, by

the factor 3.483, supposed to have an absolute error not exceeding 0.006.

We may evidently assume as a safe limit,

$$800 \times 0.006 + 3.5 \times 0.0004 = 4.8 + 0.0014$$

The product may then be made with an error not exceeding 5 simple units.

23. If we have three approximate factors, $a' = a + \alpha$, $b' = b + \beta$, and $c' = c + \gamma$, their product will be

$$a'b'c' = abc + ab\gamma + ac\beta + bc\alpha,$$

if we neglect terms containing each more than one error as a factor. The absolute error of the product will then be, very nearly,

$$a'b'c' - abc = ab\gamma + ac\beta + bc\alpha, \qquad (9)$$

and if in practice we substitute in the right hand member of this equation superior limits of the approximate quantities, we may evidently take for the limit of absolute error of the product of three approximate factors the sum of the products obtained by multiplying the error of each factor by the product of superior limits of the other two factors. The method may evidently be extended to any number of factors. *Hence, we may take for the limit of the absolute error of the product of any number of approximate factors the sum of the products obtained by multiplying the absolute error of each factor by the continued product of superior limits of all the other factors.* If any of the products $ab\gamma$, etc., would be of a much lower order than the highest, such may evidently be disregarded.

24. The formula, $a'b' - ab = a\beta + b\alpha$, assumes that the product of the approximate factors, $a'b'$, is to be exactly formed. If the product is made by

abridged multiplication the errors of the process must be allowed for. But it is always easy to reduce the error resulting from the process of abridged multiplication to very much less than that due to the errors of approximate factors.

PROBLEM II. To determine the necessary arrangement of approximate factors for abridged multiplication, so that the error due to the abridged process shall be less than that due to the error of either factor.

RULE II. *When the absolute error of the factor having the greater relative error equals or exceeds* 5 *of its lowest units, take either factor for the multiplier, reverse the order of its figures, and place it under the multiplicand as far to the right as possible without having any of the significant figures of the multiplier to the right of the multiplicand or any significant figures of the multiplicand to the left of the multiplier. When the error of the factor having the greater relative error is between* $\frac{1}{2}$ *and* 5 *of its lowest units, put the reversed multiplier one place further to the right.*

The reason for the rule will be clear from an example or two. We shall hereafter indicate the limit of absolute error of approximate factors by writing it in a special style of figure, following the factor, with the plus or minus sign, this limit of error being understood to be so many units of the lowest order retained in the factor. In reversing the multiplier its limit of error will come at the left. Where simply the plus or minus sign follows a number the error of the number will be understood to be not greater than half a unit of the lowest order retained in the number. The approximate factors of Example 18 will then be written, $784.2817 \pm \mathit{4}$ and $3.483 \pm \mathit{6}$. If we form their

product by arranging them according to Rule 11, we shall have

$$
\begin{array}{rrl}
 & 3.483 & \pm\ 6 \\
4\ \pm & \underline{7182487} & \\
 & 24381 & \pm\ 42 \\
 & 2786 & \pm\ 6 \\
 & 139 & \pm \\
 & 7 & - \\
 & \underline{3} & - \\
 & 2731.6 & \pm\ \overline{50}
\end{array}
$$

By this operation it is evident that the errors of the first and second partial products due to the error of the multiplicand amount to about 47 units of the lowest order in the result, while the errors due to the neglected parts of the last four partial products do not exceed 2 of these units. And it is evident that if the right hand figure of the reversed multiplier had been the smallest possible, viz.: 1, and the error of the multiplicand as large as 5 of its lowest units, the error of the result due to that of the multiplicand would have been at least 5 of the lowest units of the result; which would allow for 10 partial products besides the first, with errors all in the same direction, before the sum of the errors of the abridged process would equal that due to the error of the multiplicand. The result would not be essentially different if the former multiplicand were made the multiplier. For by Rule 11 we should have

$$
\begin{array}{rrl}
 & 784.2817 & \pm\ 4 \\
6\ \pm & \underline{3843} & \\
 & 23528 & + \\
 & 3137 & + \\
 & 627 & + \\
 & \underline{24} & \pm\ \underline{48} \\
 & 2731.6 & \pm\ 50
\end{array}
$$

and here the possible error of the last partial product is nearly all due to that of the same factor which caused the greatest part of the error in the former operation.

But let us look at one more example. Take the factors 1124.267543 ± 2, and 8425.7987 ± 2. If we were to make their product by arranging them like this,

$$\begin{array}{r} 8425.7987 \ \pm 2 \\ 2 \pm \underline{345762\,4211} \end{array}$$

it is evident that the amount of the error of their product due to the errors of the factors would be only about 2 of the lowest units obtained, whereas the errors of the product arising from the abridged process would be those due to the parts rejected from eight partial products; the limit of the errors arising from the abridged process could not then be placed in advance at less than 4 of the lowest units of the product. To reduce the errors due to the abridged process of multiplication to less than those due to the errors of the factors, we need therefore, in accordance with Rule II, to move the multiplier one place to the right. And this will be amply sufficient. For if we arrange the factors thus,

$$\begin{array}{r} 8425.7987 \ \pm 2 \\ 2 \pm \underline{34576\,24211} \end{array}$$

it is evident that the error of the product due to those of the factors will be about 24 of the lowest units of the product, while that due to the neglected parts of partial products will be less than 5 of the same units. In this example the highest significant figure of the multiplier is the smallest possible, which reduces the part of the error of the product due to that of the

multiplicand to a relatively small amount. And it is clear that if the limit of absolute error of the factor having the greater relative error were but half a unit of its lowest order we might add a zero to the factor, and call the error 5 units of the lowest order then retained, which would bring the case under the first part of Rule 11, already illustrated. If the relative errors of two factors are the same, or nearly the same, either factor may for the purposes of the rule be assumed as having the greater relative error. The rule is sufficient in all cases in which not more than ten partial products have errors in one direction.

EXAMPLE 19. Multiply 3.14278 ± by 0.00742 ±, and let the error from the abridged process be small in comparison with the limit of error of the product due to that of either factor.

Annexing a zero to each factor as just suggested, and then arranging by Rule 11, we have

$$\begin{array}{rrl}
 & 3.142780 & \pm\ \mathit{5} \\
\mathit{5}\ \pm & \underline{0\ 24700.0} & \\
 & 21\ 999 & + \\
 & 1\ 257 & + \\
 & 63 & - \\
 & \underline{0} & \pm\ \underline{\mathit{16}} \\
 & 0.023319 & \pm\ \mathit{17}
\end{array}$$

The error due to the abridged process is less than a tenth of the possible error of the product from that of the multiplier.

EXAMPLE 20. Multiply 4.725 ± *3* by 0.1478 ± *7* and assign a limit of error to the result.

25. PROBLEM 12. Two or more factors being given, each to a certain degree of approximation, it is required to assign a superior limit to the relative error of their product.

RULE 12. *Take for the limit of the relative error of the product, the sum of the superior limits of the relative errors of the factors.*

Demonstration. We have for two factors the very nearly exact relation, equation (8),

$$a'b' - ab = a\beta + b\alpha.$$

Dividing both members by ab we obtain

$$\frac{a'b' - ab}{ab} = \frac{\beta}{b} + \frac{\alpha}{a}.$$

The first member of this equation is by definition the relative error of the product $a'b'$, while the second member is the sum of the relative errors of the factors a' and b'; and since by the rule we take instead of these latter errors superior limits of them, it is evident that Rule 12 will give a safe limit of the error of a product of two factors. It follows then that we may take for the limit of the relative error of the product of any number of approximate factors, the sum of superior limits of the relative errors of the factors. For if the product of two factors be regarded as a single factor, the limit of its relative error plus that of a third factor may be taken for the limit of the relative error of the product of the first product by the third factor, and so on.

Rule 12 assumes, like Rule 10, that the product of the factors is to be exactly formed. If made by the abridged process the new errors introduced must be allowed for. But it has been shown that the additional absolute error introduced by the abridged process may easily be made much less than that due to the errors of the factors; hence the additional relative error due to the abridged process may also be correspondingly reduced. By placing the multiplier one

or two places farther to the right than required by Rule 11 the error from the abridged process will be quite insignificant compared with that due to the errors of the factors. By indicating the limit of error of each partial product we may also if we please determine the limit of error of the actual result independently of any rule.

EXAMPLE 21. Determine by Rule 12 the limit of relative error in the product

$$(65.432 \pm \mathit{2})\ (6.2124 \pm \mathit{2})\ (1.5632 \pm \mathit{2}),$$

and then compute the product.

The sum of the limits of the relative errors may evidently be taken at

$$\frac{2}{60000} + \frac{2}{60000} + \frac{2}{15000} = \frac{12}{60000} = \frac{1}{5000}.$$

Arranging the first two factors by Rule 11 we have

$$\begin{array}{rrcr}
 & 65.432 & \pm & \mathit{2} \\
\mathit{2}\ \pm & 4\ 212.6 & & \\
\hline
 & 392\ 592\ 0 & \pm & \mathit{120} \\
 & 13\ 086\ 4 & \pm & \mathit{4} \\
 & 654\ 3 & + & \\
 & 130\ 9 & - & \\
 & 26\ 2 & \pm & \mathit{131} \\
\hline
 & 406.489\ 8 & \pm & \mathit{256}
\end{array}$$

Call this result $406.490 \pm \mathit{26}$, and make the next product as follows:

$$\begin{array}{rrcr}
 & 406.49\ 0 & \pm & \mathit{26} \\
\mathit{2}\ \pm & 23\ 65.1 & & \\
\hline
 & 406.49\ 0 & \pm & \mathit{26} \\
 & 203\ 24\ 5 & \pm & \mathit{13} \\
 & 24\ 38\ 9 & \pm & \mathit{2} \\
 & 1\ 21\ 9 & - & \\
 & 8\ 1 & \pm & \mathit{82} \\
\hline
 & 635.42\ 4 & \pm & \mathit{124}
\end{array}$$

The limit of absolute error of the result, as indicated by the limits of error of the partial products, is 124 of the lowest units of the result; hence the relative error of this result cannot exceed $\frac{124}{630000}$, which is a trifle less than the limit determined above, viz.: $\frac{1}{5000}$.

EXAMPLE 22. Determine by Rules 10 and 12 the absolute and relative errors to be expected in the product of the following approximate factors, and then form the product and determine its error by inspection, as illustrated in example 21:

$$(756.82 \pm 3)\ (25.41 \pm 1)\ (0.3248 \pm 2).$$

26. *It follows from Rule* 10 *or equation* (8) *that the limit of the absolute error of the square of an approximate number may be taken equal to twice a superior limit of the number, multiplied by the absolute error of the number.* For if in equation (8) we make $b' = a'$ and $\beta = \alpha$ we have

$$a'^2 - a^2 = 2a\alpha. \qquad (10)$$

And it follows from this that if the limit of absolute error of an approximate number be a unit of the nth order counting down from the highest significant figure inclusive, the absolute error of the square of the number will not exceed a unit of the $(n - 1)$th order, counting down in the square also from its highest significant figure inclusive. For, supposing, as we may for the present purpose, the number a' to be entire, and to contain n figures and be correct within a simple unit, we shall have $\alpha = 1$; and a'^2 will have either $2n$ or $2n - 1$ figures. But if a'^2 has only $2n - 1$ figures, the highest significant figure of a' must be less than 4, in which case $2a\alpha$, or the absolute error of the square, will have only n figures. Hence, if $2a\alpha$ be placed under a'^2 so that the lowest units of

the two come in the same vertical line, there will be $n - 1$ figures of a'^2 at the left of $2a\alpha$. Thus, suppose $a' = 31572$ and $\alpha = 1$. We have

$$\begin{aligned} a'^2 &= 996791184 \\ 2a\alpha &= 63144 \end{aligned}$$

Here, n being five, the error of the square is less than a unit of the fourth order, counting from the left in the value of a'^2. If the first figure of a' had been 5 or greater, $2a\alpha$ would have had one more figure, but so also would a'^2, hence the same statement would hold; and the truth of our proposition is evident. And since, if a remain constant, $2a\alpha$ will be proportional to α, we may also state the principle that if an approximate number be exact within *half* a unit of the nth order from its highest significant figure, the square of the number will be exact within *half* a unit of the $(n - 1)$th order from its highest significant figure. It is evident that the principles of this paragraph assume that the square of the approximate number is to be precisely formed. If it be made by abridged multiplication, care must be taken that the errors due to the abridged process are made insignificant in comparison with $2a\alpha$; and this is in practice always easy. The principle stated at the head of this article, or equation (10), will generally give a somewhat smaller limit of error than the other principles of this article, though it is convenient to remember that we may expect in the square of an approximate number as many exact figures less one, as there are in the number itself.

EXAMPLE 23. Compute $(0.0080715 \pm 1)^2$ by abridged multiplication, and determine the limit of error of the result.

27. From the approximate value of the absolute error of the product of three factors, equation (9),

Art. 23, by making the factors and their errors equal we obtain

$$a'^3 - a^3 = 3a^2\alpha \qquad (11)$$

and if in practical examples we take in the right hand member of this equation, instead of a, a superior limit of a', *we may evidently assume as the limit of absolute error of the cube of an approximate number, the absolute error of the number, multiplied by three times the square of the superior limit of the number.* It is hardly necessary to say that if the cube is formed by abridged multiplication the new error introduced must be added, or else made insignificant.

EXAMPLE 24. Determine the limit of absolute error of $(3.456 \pm 2)^3$ and $(883.4 \pm 4)^3$, and perform the multiplications.

For the absolute error of the first of these cubes we may take

$$3a^2\alpha < 3(3.5)\,(3.5)\,(0.002) < 0.075.$$

28. By dividing both members of equation (10) by a^2, and those of equation (11) by a^3 we have the approximate equations

$$\frac{a'^2 - a^2}{a^2} = 2\frac{\alpha}{a} \qquad (12)$$

$$\frac{a'^3 - a^3}{a^3} = 3\frac{\alpha}{a} \qquad (13)$$

from which it is evident that we may take for the limits of the relative errors of the square and cube of an approximate number respectively twice and three times the limit of the relative error of the number, a result evidently in harmony also with Rule 12.

EXAMPLE 25. Apply the principle just stated, to determine the limits of the relative errors of the re-

sults of Examples 23 and 24, and from the relative errors thus found deduce also the absolute errors.

For the relative error of the first cube in Example 24 we have $3\frac{\alpha}{a} < \frac{6}{3400}$, and since the cube will not exceed 42 we may take for the limit of absolute error, $\frac{6 \times 42}{3400}$, which is the least trifle less than 0.075, the limit determined in Example 24.

It evidently follows from Rule 12, that we may take for the limit of the relative error of the n*th power of an approximate number,* n *times a limit of the relative error of the number.* If n were extremely large, however, we should not assume the limit of the relative error of the number at its smallest possible value.

DIVISION.

29. Problem 13. It is required to state a rule for determining the order of units of the highest significant figure of a quotient.

Rule 13. *Observe how many places to the right or left the decimal point of the dividend would have to be moved in order that the first figure of the quotient should be simple units. If the point would have to be moved* n *places towards the right the first significant figure of the true quotient will be in the* n*th decimal place. If the point would have to be moved* n *places towards the left the first figure of the true quotient will be* n *places above simple units; that is, the true quotient will have* $n + 1$ *figures at the left of the decimal point.*

The rule may be illustrated by an example or two. In what follows we shall arrange quantities in division as we arrange them in algebra, viz.: with the divisor at the right of the dividend, and the quotient below the divisor. Suppose then we have for division

$$\begin{array}{r|c} 0.1941690 & 763.05436 \\ \cline{2-2} & 2. \\ & 0.0002 \end{array}$$

It is plain that if the decimal point of the dividend were moved four places to the right the first figure of the quotient would be 2 simple units. If we correct that quotient by moving the decimal point back four

places to the left, the 2 will evidently be brought into the fourth decimal place. If the dividend and divisor were as follows:

$$\begin{array}{r|l} 7428.436 & \underline{0.04269} \\ & 1. \\ & 1\ 74008.8 \end{array}$$

it is clear that if the decimal point of the dividend were moved five places to the left the first figure of the quotient would be simple units. If we correct that quotient by moving its decimal point five places to the right the true quotient will evidently have six figures at the left of the decimal point, the results in each of these illustrations agreeing with the rule.

30. PROBLEM 14. A dividend and divisor being proposed, whose values may be taken as accurately as we please, it is required to form their quotient so that its absolute error shall not exceed a unit of the *n*th order from the decimal point.

RULE 14. *Begin by determining the first significant figure of the quotient, and the order of its units. Then count the number of figures that the quotient must contain, from this figure inclusive down to the* n*th inclusive. Beginning at the left, assume in the divisor a number of significant figures greater by one than the number just counted, and take at the left of the dividend as many significant figures as there are in the product of this assumed divisor by the first significant figure of the quotient. Subtract this product from the assumed dividend, and instead of annexing any figure to the remainder, reject a figure at the right of the divisor to determine the second figure of the quotient. Continue the process of division by throwing off successively the figures of the divisor, until the quotient contains one figure below the* n*th order. In multiplying the portion of the divisor each time retained, by the successive quotient figures, have regard*

to the rejected part of the divisor so far as to make each partial product if possible exact within half of one of its lowest units. Whether the extra figure of the quotient can be discarded without passing the assigned limit of error must be determined by the special conditions of each example.

EXAMPLE 26. Compute the expression

$$\frac{194\,16.9063468085\ldots}{763.05\,403678956\ldots},$$

with an absolute error in the quotient less than a unit of the 6th decimal place.

We see that the decimal point in the dividend would have to be moved one place to the left for the first figure of the quotient to be simple units, hence there will be two entire figures in the quotient. Therefore we need nine figures of the divisor and ten of the dividend. We take then

$$\begin{array}{r|l}
19416.90635 & 7\dot{6}\,\dot{3}.\dot{0}\dot{5}\dot{4}\dot{0}\dot{3}\dot{7} \\
\underline{15261\,0807.4} & 25.4\,463058 \\
4155\,82561 & \\
\underline{3815\,27018} & \\
340\,55543 & \\
\underline{305\,22161} & \\
35\,33382 & \\
\underline{30\,52216} & \\
4\,81166 & \\
\underline{4\,57832} & \\
23334 & \\
\underline{22892} & \\
442 & \\
\underline{382} & \\
60 &
\end{array}$$

We mark with a dot each figure when it is rejected from the divisor. Let us consider the possible errors introduced in this example. The error of the dividend employed is less than half a unit of the lowest order retained in it; the error of the assumed divisor being also less than half a unit of its lower order, the error of the first partial product cannot exceed a unit of the lowest order in that, since the first quotient figure, by which we multiply, is 2. If the direction of the error of the divisor were not known, the limit of error of the second partial product would also be a unit of its lowest order, since it would be doubtful whether we ought to carry a 3 or a 4 to its lowest figure, from the product of the rejected figure 7 of the divisor by the second figure of the quotient. As for the remaining partial products, none of their errors can be more than half a unit of their lowest order. But the remainder, 60, by which the 7th decimal of the quotient is determined, cannot be in error by more than the sum of the errors of the dividend and the partial products preceding this remainder, that is, $\frac{1}{2} + 1 + 1 + \frac{5}{2} = 5$. If the true remainder at the foot ought then to be 55 instead of 60, the 7th decimal of the quotient would be 7, but should be followed by other figures; but if the true remainder ought to be 65, the 7th decimal of the quotient would still be 8, though followed by other figures. Therefore the 7th decimal of the quotient actually obtained cannot be wrong by more than a unit of its own order. Hence, *a fortiori*, the error of the quotient is less than a unit of the 6th decimal place, as required. In fact, from the limit of error now determined, we may reject the 7th decimal of the quotient, either increasing the previous one or not, without passing the assigned limit of error.

31. It is plain that if the left hand significant figure of a divisor be the smallest possible, viz: 1, and it be followed by one or more zeros, then when the division has been continued until all the figures of the divisor but the 1 have been rejected, and we are ready to find the next figure of the quotient, an error in the remainder to be used in determining this quotient figure, equal to m units of the order of the remainder, will cause an error in the quotient also equal to m units of the order of the quotient figure to be found. But by the application of Rule 14 this quotient figure will be of the order next below the nth. In the most unfavorable case possible, the first figure of the quotient will be the largest possible, viz: 9; for then the limit of error of the first partial product may be $4\frac{1}{2}$ units of its order, that of the second partial product 1 unit of the same order. If we add to these the possible error of the dividend, $\frac{1}{2}$ a unit of the same order, we have still room for 8 more partial products with errors of $\frac{1}{2}$ a unit each, all in the same direction, before the error of a remainder shall equal 10 units of its lowest order. Hence, if Rule 14 is followed, the error of the quotient cannot exceed 10 units of the order next below the nth, or 1 unit of the nth order, except there should be more than 10 partial products having errors in one and the same direction, a case evidently not often likely to occur in practice. And it is clear that if the figure of the divisor following the last one to be used by the rule is known correctly, we may reduce the error of the first partial product to less than a unit of its lowest order; and then, if the number of partial products is less than 10, the error of a quotient found by the rule will in such cases be less than *half* a unit of the nth order; so that if we wish to reject the

extra figure of the quotient we can if necessary adjust the previous figure so that the error of the quotient shall still be less than a unit of the nth order.

It is not necessary to state a special rule for fixing the number of decimals to be computed in a proposed dividend and divisor so that a quotient shall be correct within a unit of an assigned order; for the application of Rule 14 is sufficient to determine the required number.

EXAMPLE 27. Compute the expression $\frac{\pi}{\sqrt{2}}$, with a relative error less than $\frac{1}{10000}$.

The quotient will be greater than 2, hence, by Rule 4, we may make an absolute error $= 0.0002$. We shall be safe then if we apply Rule 14 as if to find a quotient with a limit of absolute error equal to a unit of the 4th decimal place. We need then, by Rule 14, 6 figures each in the divisor and dividend. The division will be as follows:

$$\begin{array}{r|l}
3.14159 & 1.\dot{4}\dot{1}\dot{4}\dot{2}\dot{1} \\
\underline{2\,82842} & \overline{2.22145} \\
31317 & \\
\underline{28284} & \\
3033 & \\
\underline{2828} & \\
205 & \\
\underline{141} & \\
64 & \\
\underline{57} & \\
7 &
\end{array}$$

There are only 5 partial products preceding the last remainder, and the error of this cannot exceed 4

units of the same order, hence the error of the quotient must be less than 3 units of the order of its figure 5. If we reject the 5, the answer will be 2.2214, with an absolute error less than 0.0001, and a relative error less than $\frac{1}{20000}$, and, *a fortiori*, less than $\frac{1}{10000}$.

EXAMPLE 28. Compute the expression $\frac{\sqrt{2}}{\pi}$, with a relative error in the quotient less than $\frac{1}{10000}$.

EXAMPLE 29. Compute the expression

$$\frac{0.54674321\ldots}{\sqrt{0.0003}},$$

with a relative error in the quotient less than $\frac{1}{2500}$.

32. PROBLEM 15. A dividend and divisor being given, each to a certain degree of approximation, it is required to assign a superior limit to the absolute error of their quotient.

RULE 15. *Multiply a superior limit of the quotient by the relative error of the divisor, divide the absolute error of the dividend by an inferior limit of the divisor, and take the sum of the two results for the required limit.*

Demonstration. It is evident that the most unfavorable case, or that in which the error of the quotient will be the greatest, is that in which the errors of the dividend and divisor are in opposite directions; for example, the dividend too large and the divisor too small. Suppose then that a and b represent an exact dividend and divisor, a' and b' the corresponding approximate dividend and divisor, and α and β their absolute errors. In the most unfavorable case we have

$$\frac{a'}{b'} = \frac{a + \alpha}{b - \beta},$$

The absolute error of the quotient $\frac{a'}{b'}$ would then be

$$\frac{a+\alpha}{b-\beta} - \frac{a}{b} = \frac{ab+b\alpha-ab+a\beta}{b(b-\beta)} =$$

$$\frac{b\alpha+a\beta}{bb'} = \frac{a}{b'}\cdot\frac{\beta}{b} + \frac{\alpha}{b'}. \qquad (15)$$

But Rule 15 will evidently give a superior limit of the last member of this equation. Hence it is a safe rule.

EXAMPLE 30. Determine the limit of absolute error of the quotient $\frac{4237.5 \pm 5}{85.846 \pm 4}$.

By Rule 15 we may evidently take

$$\frac{4500}{80} \times \frac{0.004}{80} + \frac{0.5}{80} = \frac{1.45}{160} < 0.01.$$

EXAMPLE 31. Assign limits to the absolute errors of the following indicated quotients:

$$a \quad \frac{25.342 \pm 6}{0.47328 \pm 7},$$

$$b \quad \frac{334.725 \pm 3}{8874},$$

$$c \quad \frac{9432}{784.3 \pm 4}.$$

In the second of these expressions the error of the denominator is supposed to be 0, and in the last the error of the numerator is supposed to be 0.

33. It is assumed in Rule 15 that the division of the approximate numbers will be exactly made, or, at least, that if the abridged process of division is em-

ployed, care will be taken that the errors arising from the contractions of the process shall be very small in comparison with those due to the errors of the dividend and divisor.

PROBLEM 16. A dividend and divisor being given, each to a certain degree of approximation, it is required to state a rule for dividing by the abridged process, so that the error of the quotient due to this process shall be less than that due to the errors of the quantities.

RULE 16. *Begin the division as usual. Then, if the sum of the errors of the dividend and of the first partial product formed does not exceed as many units of the lowest order employed in the dividend as half the number of partial products to follow, add places to the dividend and divisor until it does.*

EXAMPLE 32. Calculate by the abridged process the quotient

$$\frac{4.35278 \pm}{3.14159 \pm},$$

and let the error from the abridged process be less than that due to the limits of error of the quantities.

$$\begin{array}{rl|l}
4.352780 & \pm 5 & 3.1\dot{4}\dot{1}\dot{5}\dot{9}\dot{0} \pm 5 \\
\underline{3\ 141590} & \underline{\pm 5} & \overline{1.385534 \pm 5} \\
1\ 211190 & & \\
\underline{942477} & \pm 2 & \\
268713 & & \\
\underline{251327} & \pm & \\
17386 & & \\
\underline{15708} & - & \\
1678 & & \\
\underline{1571} & - & \\
107 & & \\
\underline{94} & + & \\
13 & \pm \overline{14} &
\end{array}$$

By thus adding a zero to the dividend and divisor we make the limit of error of the dividend and first partial product, due to the limits of error of the quantities, 10 units of the lowest order then in the dividend, while the only partial products which are in error from the abridged process are the last four, and the sum of the errors of these does not exceed 2 units of the same order as before, hence the error of the quotient due to the use of the abridged process is much less than that due to the limits of error of the quantities.

34. We have illustrated in the above example a method by which the limit of the absolute error of an approximate quotient may be determined by an inspection of the work. By indicating the error of the dividend and of each partial product, it is evident that the sum of all these errors cannot exceed 14 of the lowest order of units in the dividend actually employed. Therefore, since the first figure of the divisor is 3, the error of the quotient cannot exceed 5 units of the order of the figure 4 of the quotient, determined by using the last remainder, 13.

EXAMPLE 33. Perform the division of Example 30, having regard to Rule 16, and determine the limit of error by inspection.

$$
\begin{array}{r|l}
4237.5 \ \pm 5 & 8\dot{5}.\dot{8}\dot{4}6 \ \pm 4 \\
\underline{3433\,8 \ \pm 2} & \overline{49.36 \quad \pm 2} \\
803\,7 \quad\ \ & \\
\underline{772\,6 \ \pm} \quad & \\
31\,1 \quad\ \ & \\
\underline{25\,8 \qquad} & \\
5\,3 \ \pm 8 &
\end{array}
$$

It will be noticed that after finding what the last figure of the quotient is to be we do not write the corresponding partial product; but we merely consider what the quotient figure in the place just found would be if the remainder by which it is determined were diminished or increased by the possible error of that remainder. Thus in the above example, if the remainder 53 were diminished by 8, the 4th quotient figure would be 5 instead of 6; and if the remainder 53 were increased by 8, the 4th quotient figure would be 7; but without a closer determination of the limits of error of the partial products we should have to assume that the 7 might be followed by other figures. Therefore if we take 6 for the fourth figure of the quotient, we place the limit of error at 2 units of its order, so as not to understate the error, though in reality we can see that the error could not be more than a trifle over 1 unit of that order; which is the limit determined by Example 30, supposing the division were to be made exactly.

EXAMPLE 34. Calculate the expression

$$\frac{\pi}{2.74384 \pm 1},$$

having regard to Rule 16.

EXAMPLE 35. Perform the divisions of Example 31, having regard to Rule 16, and determine the limits of error by inspection; then compare these limits with those found by Rule 15.

35. PROBLEM 17. A dividend and divisor being given, each to a certain degree of approximation, it is required to assign a superior limit to the relative error of their quotient.

RULE 17. *Take for the limit of the relative error of the quotient, the sum of the superior limits of the relative errors of the dividend and divisor.*

Demonstration. In the exact expression, equation (15), for the absolute error of $\frac{a'}{b'}$, viz.: $\frac{a}{b'} \cdot \frac{\beta}{b} + \frac{\alpha}{b'}$, we shall make but a very slight error if we substitute b for b'. Making the substitution, and dividing by the expression for the true quotient, we have the relative error of $\frac{a'}{b'}$, very nearly,

$$\frac{\frac{a}{b} \cdot \frac{\beta}{b} + \frac{\alpha}{b}}{\frac{a}{b}} = \frac{\beta}{b} + \frac{\alpha}{a}. \tag{16}$$

Rule 17 will give a superior limit of the right hand member of this equation. Hence it is a safe rule.

36. It is assumed in Rule 17, as in Rule 15, that the division of the approximate numbers is to be exactly performed, or that the additional error from the abridged process is to be very small in comparison with that due to the error of the dividend or divisor. But we have shown in Rule 16 how to make the error from the abridged process less than that due to the errors of the quantities; and it is evident that we may in any case make the error of the abridged process quite insignificant by adding to the dividend and divisor one or two more places than required by Rule 16.

EXAMPLE 36. Determine by Rule 17 the limit of relative error of

$$\frac{78.425 \quad \pm 7}{10.3864 \ \pm 9},$$

and from the relative error and a limit of the quo-

tient find, by Rule 2, a limit to the absolute error. Then make the division by Rule 16, and determine the error by inspection.

EXAMPLE 37. Apply the same process to the quantities in examples 30 and 31.

37. Besides the general abridged process of division already explained, there is a special formula which may sometimes be used with advantage when the divisor is but little greater or less than unity. Suppose it to be required to compute the quotient $\frac{a}{1+x}$, a being any number, and x a small fraction. Finding two terms of the quotient by algebraic division, and adding the indicated quotient of the remainder, we may write

$$\frac{a}{1+x} = a - ax + \frac{ax^2}{1+x} = a(1-x) + \frac{ax^2}{1+x}.$$

If we neglect the last term of this expression, and take $a(1-x)$ for the quotient we shall make an absolute error equal to $\frac{ax^2}{1+x}$. The corresponding relative error will be

$$\frac{\frac{ax^2}{1+x}}{\frac{a}{1+x}} = x^2$$

Now when x is very small, x^2 will be very much smaller, and in such cases we may take $a(1-x)$ as the approximate value of $\frac{a}{1+x}$. By doing so we substitute a multiplication for a division; and we may if we please easily determine the limit of absolute error

committed, knowing that the approximate result is too small, and its relative error equal to x^2.

EXAMPLE 38. Compute by the above method the expression $\frac{2}{1.0077}$, and determine the limit of error committed.

For $\frac{2}{1.0077}$ we substitute $2(0.9923) = 1.9846$. The relative error of this being $(0.0077)^2 < 0.00006$, the absolute error cannot be greater than $2(0.00006) = 0.00012$, (Rule 2); that is, the result 1.9846 is toc small by a little more than a unit of the 4th decimal place.

By treating the fraction $\dfrac{a}{1-x}$ in the same way as we treated $\dfrac{a}{1+x}$, we may also deduce

$$\frac{a}{1-x} = a(1+x) + \frac{ax^2}{1-x}.$$

If we take $\dfrac{a}{1-x} = a(1+x)$, we make an absolute error equal to $\dfrac{ax^2}{1-x}$ and a relative error equal to

$$\frac{\dfrac{ax^2}{1-x}}{\dfrac{a}{1-x}} = x^2,$$

this error being in the same direction as before, and having the same expression.

EXAMPLE 39. Divide 2 by 0.9923.

We substitute $2(1.0077) = 2.0154$; the limit of the relative error of this result being, as in the last example, 0.00006, and the absolute error in fact $< 0\ 00012$, though if the relative error were quite up to 0.0006

the absolute error would slightly exceed 0.00012, since the quotient is greater than 2.

It is evident then that where x is a small fraction, in place of $\frac{a}{1+x}$ we may substitute $a(1-x)$, and for $\frac{a}{1-x}$ we may substitute $a(1+x)$, and the results will in each case be a little too small, the relative error being x^2, and the absolute error being the quotient multiplied by x^2.

EXAMPLE 40. At a temperature, $t = 5°$ centigrade, and a pressure of 1 atmosphere, the volume, V, of a mass of hydrogen being 1 litre, what will be its volume, V_0, at zero, under the same pressure?

We have from Physics the formula

$$V_0 = \frac{V}{1+t(0.0036613)}$$

or $$V_0 = \frac{1}{1+5(0.0036613)} = \frac{1}{1+0.0183065}.$$

Say $V_0 = 1 \times (1-0.0183065) = 0.9817$ very nearly. The relative and absolute errors of this result being a little less than $(0.02)^2 = 0.0004$, we may be sure that the answer within a unit of the 4th decimal place is 0.9820.

EXAMPLE 41. Suppose the volume of the gas at a temperature of $-5°$ to be 4 litres, what will be its volume at zero? In the formulas of example 40, make $t = -5$, and $V = 4$.

SQUARE ROOT.

38. PROBLEM 18. Any exact number being given, it is required to find its square root by an abridged method, but so that the absolute error of the root shall not exceed a unit of the $(2n+1)$th order, counting to the right from the highest significant figure of the root inclusive.

RULE 18. *Employ the ordinary process of extracting the square root until* n + 1 *significant figures of the root have been found. Form the next trial divisor as usual, and in forming the new dividend bring down only one new figure from the original number. Finish the work by dividing this dividend by the trial divisor just formed, contracting the latter one place at the right after each quotient figure is found, and placing the quotient figures in the root, until* n *additional figures have been found, observing not to increase the last one unless it would be followed by a figure at least as large as 7.*

EXAMPLE 42. Compute $\sqrt{10498.59325783}$, so that the result shall contain nine figures and be exact within a unit of the lowest order retained.

We put $2n+1 = 9$, whence $n+1 = 5$. We shall then find five figures by the ordinary process, and four by division.

```
          10498.59325783 ( 102.462643
          1
 202 )    0498
           404
 2044 )    94 59
           81 76
 20486 )   12 8332
           12 2916
  ....
 20492 )      54165
              40984
              13181
              12295
                886
                820
                 66
                 61
                  5
```

After finding five figures of the root, the next trial divisor, 20492, is formed as usual; but since the next figure of the root is not to be annexed to this trial divisor before multiplying, we must evidently bring down but one new figure to the dividend. In bringing down this new figure, we do not increase it, whatever it is followed by in the original number, for the contracted process tends to make the result a trifle in excess, as will be seen below. The last four figures of the root are given by division.

To show that the error of a square root found by Rule 18 is less than the limit assigned in the statement of the problem, we may proceed as follows: Since the process of extracting the square root of a number which is partly decimal does not essentially

differ from that in which the number is entire, we may confine the investigation of the principle to the extraction of the roots of whole numbers. These roots, however, may be entire, or they may be partly decimal.

Suppose the square root of an entire number to be separated into two parts, the part at the right being made to contain n figures besides the decimal figures, and the part at the left at least $n + 1$ figures. Denote the relative value of the left hand part by a, and that of the remaining part by b. (By the relative value of the left hand part we mean the value of the $n + 1$ figures with n zeros added). If N be the number whose root we are considering we shall then have,

$$N = (a + b)^2 = a^2 + 2ab + b^2. \qquad (17)$$

If we employ the ordinary process of extracting the square root of the number N, until we have found the part a, and after forming as usual the next dividend bring down to this the remaining periods of the number, this dividend will evidently be equal to $N - a^2$. Finding the value of $N - a^2$ from equation (17) and denoting it by R, we have

$$R = 2ab + b^2,$$

whence $$b = \frac{R}{2a} - \frac{b^2}{2a}. \qquad (18)$$

Let us determine a superior limit of the value of this last term, $\frac{b^2}{2a}$. Since b has but n figures above the decimal point, b^2 cannot have more than $2n$ figures above that point, whereas a, with its relative value, must have at least $2n + 1$ places above the decimal point; and $2a$ must hence be more than twice as great as b^2. Therefore the value of $\frac{b^2}{2a}$ is less than

half a simple unit. It is thus evident from equation (18) that if the dividend R were exactly divided by $2a$, and the quotient taken for the part b of the square root, the result would be too large, but by *less than half a unit.*

In employing the abridged process of division for finding the value of the quotient $\frac{R}{2a}$, the only additional error introduced is that due to the rejection of some figures from a few partial products. A limit to the sum of these errors may be determined as follows: Since the part a, without regard to its relative value, contains $n + 1$ figures, the next trial divisor, which is $2a$, will also contain at least $n + 1$ figures. Now for each of these $n + 1$ figures in the divisor a new figure of the root can be found, the last one being therefore one place to the right of the $2n + 1$ figures to be found by Rule 18. The error of the final remainder by which this extra figure would be determined, due simply to the successive cutting off of n figures from the trial divisor, will not exceed n times the half of one of the lowest order of units in this remainder. And if the divisor $2a$ does not contain more than $n + 1$ figures, the left hand one must be at least equal to 2; so that the error of the extra figure of the root due to the cause we are now considering will not exceed n times a quarter of a unit of its own order. Therefore, if the total number of figures to be found by Rule 18 do not exceed 13, n being then 6, if the redundant figure be annexed, the total error of the root will not exceed 7 units of the order of this figure, allowing 5 for the error due to disregarding the part $\frac{b^2}{2a}$ in equation (18), $1\frac{1}{2}$ for the errors of the contracted division, and $\frac{1}{2}$ for a possible error in adjusting

this extra figure with reference to what would follow. Hence, when this redundant figure would be 7 or over, if we reject it without increasing the previous figure the result will be too small, and if we reject it by adding 1 to the previous figure the result will be too large; but in neither case will the error be greater than a unit of the lowest order then retained, which will be the $(2n + 1)$th. In the great majority of examples the error of a square root found by Rule 18 will be less than *half* a unit of this order; and the direction of the error will also frequently be apparent, bearing in mind that the exact quotient $\frac{R}{2a}$ would make the result too large.

EXAMPLE 43. Compute $\sqrt{0.0006620567113}$, so that the error of the result shall not exceed a unit of the 8th decimal place.

We see that the first significant figure of the root will be in the 2d decimal place, hence the error is required to be less than a unit of the order of the 7th significant figure, giving $n + 1 = 4$.

$$\begin{array}{rl}
 & 0.0006620567113 \ (0.025730462 \\
 & \underline{4} \\
45\,) & 262 \\
 & \underline{225} \\
507\,) & 3705 \\
 & \underline{3549} \\
5143\,) & 15667 \\
 & \underline{15429} \\
5146\,) & 2381 \\
 & \underline{2058} \\
 & 323 \\
 & \underline{309} \\
 & 14
\end{array}$$

After finding four significant figures of the root, the next dividend, after annexing one new figure from the number, will not contain the trial divisor, hence a zero is placed in the root, and the next figure of the root is found by cutting off one figure of the divisor. We have found one figure more in the root than was required, and if we reject this redundant figure we shall know that the result will be correct within less than half a unit of the 8th decimal place.

EXAMPLE 44. Compute $\sqrt{2}$, so that the relative error shall be less than a ten-millionth.

The allowable absolute error may be taken

$$\frac{1.4 \times 1}{10000000} = 0.00000014,$$

and since the first figure of the root is in units place we may put $2n + 1 = 8$, whence $n + 1 = 4\frac{1}{2}$. Where $n + 1$ happens to come a mixed number like this we evidently have to take for $n + 1$ the next largest whole number, in order to bring it under the rule. Find therefore in $\sqrt{2}$ five figures by the ordinary process, and the rest by contraction.

EXAMPLE 45. Compute $\sqrt{5}$ and $\sqrt{150}$, with twelve decimals in each.

39. If it be required to extract the square root of a number equal to unity plus or minus a small fraction, we may at once write the root equal to unity plus or minus one-half the difference between that and the number; and if by this process n zeros follow the decimal point in the root, we may get the root with a limit of absolute error of half a unit of the order of the $2n$th decimal place. For if $N = (1 \pm b)^2$, then in equation (18), neglecting its last term we should have

$b = \pm \frac{N-1}{2}$. That the limit of error is what we have stated is evident. For if b is a decimal fraction having n zeros following the decimal point, b^2 would have $2n$ such zeros, and therefore $\frac{b^2}{2}$, or the neglected part of equation (18), would be less than half a unit of the $2n$th decimal place.

Thus we may write $\sqrt{1.01239} = 1.006195$, within 0.00005; and since we know from equation (18) that the error committed makes the result too large, we may take either 1.0061 or 1.0062 as the result within 0.0001.

Also we may write $\sqrt{0.9821} = \sqrt{1-0.0179} = 1-0.00895 = 0.99105$, within 0.00005; and since this result is also too large we may reject the last decimal, and the result will be 0.9910, with an error less than half a unit of its lowest order. To show that a result found like this 0.99105 is too large, observe that if $N=(1-b)^2$, then equation (18) might be written $b = \frac{1-N}{2} + \frac{b^2}{2}$, so that neglecting the last term of this equation makes b too small, and therefore $(1-b)$ too large.

EXAMPLE 46. Write the values of $\sqrt{1.01719}$ and $\sqrt{0.98911}$ within 0.00005.

40. PROBLEM 19. Any approximate number being given, with an absolute error not exceeding an assigned limit, it is required to assign a limit to the absolute error of the square root of the number.

RULE 19. *Take for the limit of the absolute error of the square root, the absolute error of the number, divided by an inferior limit of twice the said square root.*

Demonstration. Suppose that a'^2 represents any approximate number, and that $a' = a + \alpha$ is the square root of it when precisely taken, α being the error of the root a' due to that of the number a'^2. We have $a'^2 = a^2 + 2a\alpha + \alpha^2$, or, by transposing and neglecting the very small quantity α^2, we have, as in equation (10), the following nearly exact equation,

$$a'^2 - a^2 = 2\,a\alpha,$$

whence

$$\alpha = \frac{a'^2 - a^2}{2a}. \qquad (19)$$

The numerator of the last member of equation (19) is the absolute error of the number a'^2, and the equation gives a little larger value of α than the true value, hence Rule 19 will give a safe limit for α, provided the root of the approximate number is exactly taken, or that the new error introduced by the contracted method of finding the root be made insignificant, and this is in practice always easy. For, after finding by Rule 19 as small a limit as is convenient, to the error of the root if precisely taken, it will be easy to see what value $n + 1$ must have in applying Rule 18, so that the error due to the abridged process shall be less, or as many times less, as we please.

EXAMPLE 47. Given $\pi = 3.14159265$, so that the absolute error, without stating its direction, is less than half a unit of the 8th decimal place, determine by Rule 19 a limit to the error of its square root if precisely taken, and then find the root by the abridged process, but let the new error introduced by that process be less than a tenth of the limit found by Rule 19.

By Rule 19 we may take for the limit of error $\frac{0.000000005}{2 \times 1.7} < 0.0000000015$. The conditions of the

problem will then evidently be satisfied if we apply Rule 18 so that the error of the process shall be less than a unit of the 10th decimal place, and since the root has one entire figure, we have $2n+1 = 11$, and $n+1 = 6$. Finding 6 figures by the ordinary process, and 5 by division, we get $\sqrt{\pi} = 1.7724538499 \pm 16$.

41. It results from equation (19) that if the limit of absolute error of an approximate number be a unit of the nth order, counting down from the highest significant figure of the number, the square root may be found with an error not exceeding 2 units of the nth order, counting down in the root also from its highest significant figure; and unless the first significant figure of the root is 3 or 4, the limit of error may be made a *single* unit of the same order as before. For in equation (19) suppose that a^2 has n figures above the decimal point, and that $a'^2 - a^2$ equals a simple unit; then $\alpha = \frac{1}{2a}$. The largest value of α in this equation will correspond to the smallest value of a; but if a^2 has n figures above the decimal point, a will have either $\frac{1}{2}(n+1)$ or $\frac{1}{2}n$ figures above that point, and if it has only $\frac{1}{2}n$ then the highest one must be at least equal to 3; so that α will not exceed unity divided by a number containing $\frac{1}{2}n$ figures, the first of these being at least equal to 6. To take a definite case, suppose an entire number, a'^2, to contain 8 figures, and its error to be a simple unit; then $\alpha = \frac{1}{2a}$ will be less than $\frac{1}{6000} < 0.0002$. But the root will have 4 figures besides the decimals; hence its error will be less than 2 units of the 8th order from its highest figure. And if the first figure of the root in

this example were as large as 5, then $\alpha = \frac{1}{2a}$ would not exceed $\frac{1}{10000} = 0.0001$. In making n an even number we have taken the most unfavorable case. *It follows then that when an approximate number contains* n *significant figures, and is exact within a unit of its lowest order, we may find the square root with an error not exceeding a unit of the order of its* n*th significant figure, except when the first significant figure of the root is* 3 *or* 4, *and in those cases the limit of error need not exceed* 2 *units of the same order.* In the majority of cases we may find as many exact significant figures in the square root of an approximate number as there are in the number. (Compare examples 47 and 50).

42. PROBLEM 20. Any approximate number being given, with a relative error not exceeding an assigned limit, it is required to assign a limit to the relative error of the square root of the number.

RULE 20. *Take for the limit of the relative error of the square root, one-half of the limit of the relative error of the number.*

Demonstration. Dividing both members of equation (19) by a, we have the nearly exact equation,

$$\frac{\alpha}{a} = \frac{1}{2} \cdot \frac{a'^2 - a^2}{a^2}. \qquad (20)$$

The factor $\frac{a'^2 - a^2}{a^2}$ being the relative error of the approximate number a'^2, and the slight error committed in finding equation (19) being on the side of safety, Rule 20 is evidently a safe one. It is also evidently in harmony with the principles of Art. 28.

EXAMPLE 48. The absolute error of the number

85.4374 being supposed less than 4 of its lowest units, assign a limit to the relative error of its square root, and from this limit assign the limit of absolute error of the root.

It is hardly necessary to say that Rule 20 assumes that if the root is to be found by the abridged process the new error introduced will be rendered insignificant, or else allowed for.

43. Independently of Rules 18 and 19 it is easy to find by inspection of the work in any example a limit to the possible error committed in the square root. We give an example or two.

EXAMPLE 49. Compute $\sqrt{15}$.

```
        15  (3.87298335 ± 3
         9
   68 ) 600
        544
  767 )  5600
         5369
 7742 )   23100
          15484
 7744 )    76160
  ....     69704 +
            6456
            6197 −
             259
             232 +
              27
              23 +
               4 ± 2
```

After finding 4 figures of the root, and forming the next trial divisor, 7744, if, in beginning the contrac-

tion by making the partial product of this divisor by the root figure 9, we carry to that product 8 units of its lowest order, which would be carried if the 9 were annexed to the trial divisor, the partial product, 69704, will be wrong by less than half a unit of its lowest order; and if in getting the next partial product we multiply by the next root figure 8 as if the figure 4 to be rejected in the divisor were a 6, that being the nearest what it would be if the part of the root already found were doubled, the partial product 6197 is also correct within half a unit of its lowest order. The other figures of the divisor being successively rejected, the final remainder 4, by which the figure 5 of the root is determined, cannot be wrong by more than 2 units of its order; hence by the principle of Art. 34 this last figure of the root cannot be wrong by more than 3 of its units.

EXAMPLE 50. Compute $\sqrt{75.438175 \pm 2}$.

$$
\begin{array}{rl}
 & 7\dot{5}.4\dot{3}8\dot{1}7\dot{5} \pm 2 \ (8.6855152 \pm 2 \\
 & \underline{64} \\
166\,) & 11\,43 \\
 & \underline{\ \ 9\,96} \\
1728\,) & 1\,4781 \\
 & \underline{1\,3824} \\
17365\,) & \ \ 95775 \\
 & \underline{\ \ 86825} \\
17\dot{3}\dot{7}\dot{0}\,) & \ \ \ 89500 \\
 & \underline{\ \ \ 86853} \ - \\
 & \ \ \ \ 2647 \\
 & \underline{\ \ \ \ 1737} \ + \\
 & \ \ \ \ \ 910 \\
 & \underline{\ \ \ \ \ 869} \ - \\
 & \ \ \ \ \ \ 41 \pm 21
\end{array}
$$

The partial products in the contractions are treated as in example 49. Only two of these have their errors in the same direction; therefore the error of the final remainder, 41, cannot be more than 21 of its units, allowing 20 for the original error of the number. Hence the last figure which we have found in the root cannot be wrong by more than 2 of its own order of units. The limit of absolute error of the root in this example if found by Rule 19 would be $\frac{0.000002}{17} < 0.0000002$.

EXAMPLE 51. Compute $\sqrt{314.75435 \pm 6}$ so that the error of the result shall be mainly due to that of the number, and find the limit of the error by inspection of the process, as in examples 49 and 50.

CUBE ROOT.

44. PROBLEM 21. Any exact number being given, it is required to find its cube root by an abridged method, but so that the absolute error of the root shall not exceed a unit of the $(2n+1)$th order, counting to the right from the highest significant figure of the root inclusive.

RULE 21. *Employ the ordinary process of extracting the cube root until* n + 1 *figures of the root have been found. Form the next trial divisor as usual, except that no places are added to it at the right, and bring down to the new dividend no new figures from the number. Then, if the first figure of this trial divisor be less than* 5, *reject all but the* n + 2 *left hand significant figures of the divisor, otherwise all but* n + 1, *and reject from the dividend as many figures less one as are thus rejected from the divisor. Then divide by the contracted method of division, placing the quotient figures in the root, and reject all the figures of the root that would follow the* (2n + 1)*th, without increasing the last one retained, whatever be the value of the one which would follow.*

EXAMPLE 52. Compute $\sqrt[3]{25}$, so that the absolute error shall be less than a unit of the 10th decimal place.

			25(2.9240177382
			8
		1200	17000
69	621	1821	16389
18	81	252300	611000
872	1744	254044	508088
4	4	25579200	102912000
8764	35056	25614256	102457024
8	16	256493280000	454976000000
877201	877201	256494157201	256494157201
	1	2564950,34403	19848184,2799
			17954652
			1893532
			1795465
			98067
			76949
			21118
			20520
			598

The root having one entire figure, we have $2n+1=11$, whence $n+1=6$. After finding six figures of the root by the ordinary process, and forming the next trial divisor, viz., 25649503403,* observe that if the divisor were to be completed as usual, it would have but 2 places added to it, while the dividend would have 3, so that in the contracted division we must discard at first one more figure in the trial divisor than in the dividend. Rejecting, therefore, all but $n+2$ figures of the divisor, and rejecting one less in the dividend than in the divisor, we finish the work by contracted division.

* Students who are not used to finding the cube root in the way we have begun the work, will understand that this trial divisor is simply three times the square of the first six figures of the root.

To examine the limit of error of a cube root found by Rule 21, we may proceed as in the case of square root, by considering only the case in which the number is an entire number, but its root either entire or partly decimal. If N represent the number whose cube root is to be taken, b the right hand part of the root, containing n figures besides the decimal part, and a the relative value of the left hand part of the root, with at least $n+1$ figures, besides the n places corresponding to the part b, we shall have

$$N=(a+b)^3= a^3+3a^2b+3ab^2+b^3,$$

whence, $N-a^3= 3a^2b+3ab^2+b^3$.

The first member of this equation is evidently the remainder which would be left by the ordinary process of extracting the cube root, after the part a had been found, if all the remaining periods of the number were brought down to the new dividend. Putting R for $N-a^3$, and transposing and dividing, we obtain

$$b = \frac{R}{3a^2} - \left(\frac{b^2}{a^2} + \frac{b^3}{3a^2}\right) \qquad (21)$$

To determine the greatest possible value of the last term of this equation, viz., $\left(\frac{b^2}{a} + \frac{b^3}{3a^2}\right)$, observe that b^2 has at most $2n$ figures at the left of the decimal point, while a, with its relative value, has at least $2n+1$ such figures; hence, $\frac{b^2}{a}$ can never be greater than 1, and will generally be considerably less than that. Also b^3 cannot have more than $3n$ figures at the left of the decimal point, while $3a^2$ cannot have less than $2(2n+1)-1 = 4n+1$ such figures; hence $3a$ must have $n+1$ more figures than b^3 has. The greatest value of $\frac{b^3}{3a^2}$ will evidently then be when n is

the smallest possible, viz., 1. If $n=1$, the value of b cannot exceed 10, nor that of a be less than 100; hence, $\frac{b^3}{3a^2} < \frac{1000}{30000} = \frac{1}{30}$. We therefore have, in the most unfavorable case possible, $\left(\frac{b^2}{a} + \frac{b^3}{3a^2}\right) < \left(1 + \frac{1}{30}\right)$. It is easy to see, then, that in all ordinary cases the value of this whole term will be considerably less than 1. If we even make b as large as 99, and make a the smallest possible for that value, viz., 10000, it may be proved by computation that $\frac{b^2}{a} + \frac{b^3}{3a^2}$ will be less than 1.

Hence, from equation (21), if the remainder R, left after the ordinary process of finding $n+1$ figures of the cube root, were exactly divided by $3a^2$, which is the trial divisor for finding the next root figure, the quotient would be in excess of the true value of the part b of the root, but, in all ordinary cases, the error would be considerably less than a unit of the $(2n+1)$th order, counting down from the highest significant figure of the root inclusive. And if the number of figures retained in the divisor be not less than Rule 21 requires, the new error from contracting the division will be insignificant. Therefore, with the additional security that we correct a part of the error by rejecting all the figures of the root which would follow the $(2n+1)$th, whatever be their magnitude, we may accept Rule 21 as giving the result safely within the assigned limit of error.

EXAMPLE 53. Compute $\sqrt[3]{0.009}$, with an absolute error less than 0.000000001. Also $\sqrt[3]{5}$, with 10 decimals, and $\sqrt[3]{130}$, with 9 decimals.

For the first of these, make $2n+1=9$, whence $n+1=5$.

```
                                    |0.009 (0.208008382
                                    |0.008
                 |0.120000          |    1000000
0.608   4864|0.124864          |     998912
                  64|0.129792,0000|      1088000,000
                                           1038336
                                             49664
                                             38938
                                             10726
                                             10383
                                               343
```

45. If it be required to extract the cube root of a number equal to unity plus or minus a small fraction, we may at once write the root equal to unity plus or minus one-third the difference between that and the number; and if by this process n zeros follow the decimal point in the root, we may write n more figures of the root, the limit of absolute error being a unit of the order of the last of these, or at most a unit and one-thirtieth. For if $N=(1\pm b)^3$, then, in equation (21), neglecting its last term, we should have $b=\pm\frac{N-1}{3}$. That the limit of error is what we have stated is evident from the application of Rule 21. For, if we were finding as usual the cube root, say of 1.002, we should have the figure 1 of the root, and the three zeros which would follow, by the ordinary process, and by Rule 21 we could find three more figures of the root by division with the next trial divisor, which would evidently be 3, followed by zeros. If $N=(1+b)^3$, we should reject all the figures of the result after the $2n$th decimal, without increasing this, whatever it would be followed by. Thus we may write $\sqrt[3]{1.02147} = 1.0071$, within 0.0001.

We may also write $\sqrt[3]{0.97643} = \sqrt[3]{1-0.02357} = 1-0.0079 = 0.9921$, within 0.0001. Observe that if, as in this case, $N = (1-b)^3$, equation (21) would become $b = \frac{1-N}{3} + \left(b^2 - \frac{b^3}{3}\right)$, so that neglecting the last term here tends to make b too small, and therefore the final result $(1-b)$ too large; hence, when $N = (1-b)^3$, if we find in b in all $2n$ decimals, no following ones must be rejected, without increasing by 1 the last one retained in b, as just illustrated in $\sqrt[3]{0.97643}$.

EXAMPLE 54. Write the values of $\sqrt[3]{1.0157}$ and $\sqrt[3]{0.98}$ within 0.0001.

The above method may be extended to the extraction of the nth root of a number equal to unity plus or minus a small fraction; but the limit of error increases slightly as n increases.

46. PROBLEM 22. Any approximate number being given, with an absolute error not exceeding an assigned limit, it is required to assign a limit to the absolute error of the cube root of the number.

RULE 22. *Take for the limit of the absolute error of the cube root, the absolute error of the number, divided by an inferior limit of three times the square of the said cube root.*

Demonstration. Suppose that a'^3 represents any approximate number, and that $a' = a + \alpha$ is the cube root of it when precisely taken, α being the error of the root a' due to that of the number a'^3. We have

$$a'^3 = a^3 + 3a^2\alpha + 3a\alpha^2 + \alpha^3.$$

In finding the value of α from this equation, we shall make a very slight error on the side of safety, by neglecting the last two terms. We thus have, as in equation (11), very nearly $a'^3 - a^3 = 3a^2\alpha$; whence

$$a = \frac{a'^3 - a^3}{3a^2} \qquad (22)$$

The numerator of the last member of this equation is the absolute error of the number a'^3; and Rule 22 substitutes for the denominator a little smaller value; hence, if the cube root of the approximate number be exactly found, or if the error from the contracted process of finding the root be made inconsiderable, Rule 22 will give a safe limit of the total error of the root. In practice we may find, by Rule 22, as small a limit as convenient, and then easily see what value $n+1$ must have in applying Rule 21, so that the possible new error from the contracted process of finding the root shall be small in comparison with that due to the original error of the number.

EXAMPLE 55. If we take $\sqrt{26}=5.09$, we make an absolute error less than 0.01. Determine by Rule 22 a limit to the absolute error of $\sqrt[3]{5.09 \pm I}$, and then find the root by the abridged method, but so that the new error introduced shall not exceed a tenth of that due to the error of $5.09 \pm I$.

By a little trial we may see that the cube root will exceed 1.7. The square of this is 2.89, and this multiplied by 3 gives 8.67. We may take then for the required limit $\frac{0.01}{8}=0.00125$. In order to make the error from the abridged process of finding the root less than a tenth of this, we may work then as if the limit were 0.0001; and since the root will have one entire figure, we have $2n+1=5$. Find, therefore, three figures of $\sqrt[3]{5.09}$ by the ordinary method, and two by division, and the result will be the value of $\sqrt[6]{26}$, within less than $1\frac{1}{2}$ units of thousandths place.

47. It results from equation (22), that, if the limit of absolute error of an approximate number be a unit

of the nth order, counting down from the highest significant figure of the number, the cube root may be found with an error not exceeding 2 units of the nth order, counting down in the root also from its highest significant figure; and unless the first significant figure of the root is 4 or 5, the limit of error may be made a *single* unit of the same order as before. For in equation (22) suppose that a^3 has n figures above the decimal point, and that $a'^3 - a^3$ equals a simple unit, then $\alpha = \frac{1}{3a^2}$. The largest value of α in this equation will correspond to the smallest value of a. But if a^3 has n figures above the decimal point, a will have either $\frac{1}{3}(n+2)$ or $\frac{1}{3}(n+1)$ or $\frac{1}{3}n$ figures above that point, the latter being the most unfavorable case. And if a has just $\frac{1}{3}n$ figures above the decimal point, the left hand one must be as large as 4, followed by 6, since $\sqrt[3]{100} > 4.6$. To take a definite case, suppose an entire number, a'^3, to contain 6 figures, and its error to be a simple unit. Then, since the cube root a would be greater than 46, we have $3a^2 > 6000$; hence, $\alpha < \frac{1}{6000} < 0.0002$. But the root a will have 2 figures besides the decimals; hence its error will be less than 2 units of the 6th order from its highest figure. And if the first figure of the root here were as large as 6, we should have $3a^2 > 10000$, and α would therefore be less than 0.0001. *It follows, then, that when an approximate number contains* n *significant figures, and is exact within a unit of its lowest order, we may find the cube root with an error not exceeding a unit of the order of its* n*th significant figure, except when the first significant figure of the root is* 4 *or* 5, *and in those cases the limit of error need not exceed* 2 *units of the same order.* In the majority of cases we may find as many exact significant figures in the cube root of

an approximate number as there are in the number. (Compare Example 58, &c.)

EXAMPLE 56. Determine, by Rule 22, the limit of absolute error of $\sqrt[3]{0.000136425 \pm I}$, and then find the root by the abridged process, making the error of the process, however, not more than a tenth of the limit so determined. Do the same with $\sqrt[3]{365.046 \pm}$.

48. PROBLEM 23. Any approximate number being given, with a relative error not exceeding an assigned limit, it is required to assign a limit to the relative error of the cube root of the number.

RULE 23. *Take for the limit of the relative error of the cube root, one-third the limit of the relative error of the number.*

Demonstration. Dividing both members of equation (22) by a, we have

$$\frac{\alpha}{a} = \frac{1}{3} \cdot \frac{a'^3 - a^3}{a^3}.$$

The factor $\frac{a'^3 - a^3}{a^3}$ being here the relative error of the approximate number a'^3, the limit of the relative error of the cube root a' may evidently be taken equal to one-third that of the number, provided the root of the approximate number be precisely found, or the new error in finding it be made insignificant.

Rule 23 evidently agrees with the principles of Art. 28. And it is clear that we may extend the method to a root of any index, *and take for the limit of the relative error of the* n*th root of an approximate number, the* n*th part of the limit of the relative error of the number.*

EXAMPLE 57. Assign a limit to the relative error of $\sqrt[3]{842.731 \pm 6}$, and from this limit find, by Rule 2, a limit to the absolute error of the result. Also find

this latter limit by Rule 22, and compare the two answers.

49. By a method similar to that of Art. 43, we may determine the possible error of a cube root by inspecting the possible error of the last remainder.

EXAMPLE 58. Compute $\sqrt[3]{1272.4386 \pm 7}$ as closely as the limit of error allows.

Assuming that we can find eight figures in the root, we will find five before beginning the contracted division.

			1272.4386 ± 7 (10.836248 ± 3
			1
		30000	272438
308	2464	32464	259712
16	64	3499200	12726600
3243	9729	3508929	10526787
6	9	351866700	2199813000
32496	194976	352061676	2112370056
		36,35225,6688	87442,944
			70451 +
			16991
			14090 +
			2901
			2818 +
			83 ±703

The final remainder, 83, is liable to error of 700 units of its lowest order, from the original error of the number. Allowing 3 units of the same order for the rejected parts of the partial products, and adding the 83 to its possible error, 703, the quotient of the sum by 352, the last divisor employed, will evidently be but a trifle over 2 units of the same order as the last figure 8 of the root, found by the same divisor. The error due to neglecting the part of the divisor

corresponding to $\frac{b^2}{a}+\frac{b^3}{3a^2}$, is, by Rule 21, less than a tenth of a unit of the same order; hence the total error of the root can be but little over two units of that order. We call the limit 3, so as not to understate it. The limit, if found by Rule 22, would be just about 2 of these units.

EXAMPLE 59. Compute $\sqrt[3]{752.43275 \pm 8}$ as closely as the limit of error allows, and determine the limit by inspection.

4*

LOGARITHMS AND TRIGONOMETRIC FUNCTIONS.

50. If we examine a table of common logarithms, we shall notice that the greatest difference between the logarithms of any two consecutive numbers, of n figures each, is less than 5 units of the order of the nth *decimal* of the logarithm, and that the least difference of any two such logarithms is about half of one of these units. For example, the logarithms of 1000 and 1001 are respectively 3.000000 and 3.000434, while the logarithms of 9998 and 9999 are 3.999913 and 3.999957. On the other hand, the greatest difference between any two numbers corresponding to two logarithms that differ by a unit of the nth order of decimals, is less than 3 units of the order of the nth significant figure of the numbers, and the least difference of any two such numbers is about a quarter of a unit of the same order. For example, the numbers corresponding to the logarithms 3.999800 and 3.999900 are 9995.4 and 9997.7, while the numbers corresponding to the logarithms 3.000000 and 3 000100 are 1000 and 1000.23.

The limits of difference stated above do not take into account the slight inaccuracy which will sometimes occur in finding by interpolation logarithms or numbers intermediate between those given directly in the tables. But from the examples it is evident

that in finding the logarithm of an approximate number, an error in the number equal to a unit of the order of its nth significant figure will be liable to make the nth decimal of the logarithm uncertain; and an error in a logarithm equal to a unit of the nth order of decimals will be liable to make the nth significant figure of the number found from the logarithm uncertain. Hence, if computations are made with the ordinary six-figure logarithms, the result cannot in general be depended on as accurate beyond the 5th significant figure, though if the data to begin with are exact the 6th significant figure of the result may not be far out of the way.

But since it is clear from the examples given that the amount of uncertainty in the logarithm of an approximate number consisting of n figures will vary very much according to the value of the highest significant figure of the number, it is not easy to give a general rule which will always determine the smallest obtainable limit of error in the result of a computation of approximate quantities by means of logarithms. This limit is, however, easily determined in any special example by indicating in the work the possible error of each step of the computation, which may be readily ascertained by observing what change would occur in each logarithm taken from the table, for a change in the approximate number equal to its possible error, or what change would occur in a number to be found from an approximate logarithm, for a change in this logarithm equal to its possible error. The same method is evidently applicable with the logarithms of trigonometric functions.

EXAMPLE 60. Given one side of a triangle $a = 3500 \pm 1$ feet, and the adjacent angles, $B = 65°30' \pm 30''$ and $C = 85°30' \pm 30''$, compute the side b by means of logarithms, and assign the limit of error.

We have the formula $b = a\dfrac{\sin B}{\sin A}$ Observing that the angle A will be $29° \pm 1'$, and finding the logarithms, we have:

$$\begin{array}{rrl}
\log a \;.\;. & (3500 \pm \mathit{1}) \;.\;.\;. & 3.544068 \pm \mathit{125} \\
\log \sin B & (65°30' \pm 30'') \;. & 9.959023 \pm \mathit{29} \\
a.c. \log \sin A & (29° \pm 01') \;.\;. & 0.314429 \pm \mathit{229} \\
\hline
\log b & (6569.3 \pm \mathit{5.9}) \;. & 3.817520 \pm \mathit{383} \quad d = \mathit{66} \\
 & & \qquad\qquad\;\; \mathit{330} \\
\hline
 & & \qquad\qquad\;\;\; \mathit{53}
\end{array}$$

We place at the right of each logarithm the change which the tables would give, if the quantity whose logarithm we are taking were changed by the amount of its possible error. In looking out the number for b, corresponding to the sum of the logarithms, we divide the possible error of this sum, viz., 383, by the tabular difference 66, giving the possible error of b a little less than six feet.

EXAMPLE 61. Compute in the same way the side c of the same triangle.

51. We may employ a like method in working examples by means of natural trigonometric functions.

EXAMPLE 62. Find the side b, of example 60, by means of natural functions, and assign its limit of error.

We have,

$$\begin{array}{rr}
\text{nat} \sin (65°30' \pm 30'') = 0.90996 \pm & \mathit{7} \\
\text{Multiply by } a \text{ reversed, } \mathit{1} \pm \quad 0053 & \\
\hline
272988 \pm & \mathit{21} \\
45498 \pm & \mathit{4} \\
0 \pm & \mathit{91} \\
\hline
3184.86 \pm & \mathit{116}
\end{array}$$

Divide by nat $\sin(29° \pm 1') = 0{\cdot}48481 \pm\ 26$

$$
\begin{array}{r|l}
3184{\cdot}86 \pm \mathit{116} & 0{\cdot}48481 \pm \mathit{26} \\
\underline{290886} \pm \mathit{156} & \overline{6569{\cdot}3 \pm \mathit{6{\cdot}0}} \\
27600 \qquad & \\
\underline{24241} \pm \quad \mathit{16} & \\
3359 \qquad & \\
\underline{2909} \pm \underline{\quad \mathit{2}} & \\
450 \quad \mathit{290} & \\
\underline{436} \qquad & \\
14 \qquad &
\end{array}
$$

When we reach the remainder 450, by which the units figure of the quotient is obtained, we see that the possible error of this remainder has amounted to 290 units of its lowest order. Dividing this possible error by the same portion of the divisor that was used in finding the units figure of the quotient, we have, as the limit of error of the result, 6 feet, which only differs by a tenth of a foot from the limit of error when the computation was made by logarithms, while the result itself is the same. It is thus seen that natural functions with five decimals give nearly the same precision as logarithmic functions with six decimals.

EXAMPLE 63. Compute in the same way the side c of the same triangle, and compare with the result of example 61.

COMPLEX COMPUTATIONS.

52. Having explained the elementary processes of approximate computations, we will now consider the case in which several of these processes are to be employed in a single problem. We will first take examples in which the quantities proposed may be found as accurately as we please, and afterwards give a few in which some of the quantities are known with only a limited degree of accuracy.

PROBLEM 24. Any complex monomial being proposed, whose value is required with an absolute error not exceeding an assigned limit, and whose factors may each be taken as accurately as we please, it is required to assign an allowable limit to the error of each factor, and to compute the value of the monomial.

RULE 24. *Make a rough calculation of a superior limit of the result, and from this determine, by Rule* 3, *the allowable relative error of the result. Then count the number of single factors whose values are to be taken approximately, both in the numerator and denominator of the monomial, and add to the number of such factors the number of operations that are to be performed on them after their values are obtained; divide the allowable relative error of the final result by the sum thus made, and the quotient will be an allowable relative error for each single factor. Compute each single factor until its error does not exceed this limit, and then perform the remaining operations in such a way that the new error introduced by each abridged process shall be very small compared with that due to the errors of the factors. Indicate, as the work proceeds, the limit of error of each partial result, and lastly of the final result.*

EXAMPLE 64. Compute the value of

$$\frac{\sqrt[3]{2} \times \sqrt{\dfrac{\sqrt[3]{10} \times \pi}{5.27963289\ldots}}}{0.4318965021\ldots}$$

so that the absolute error shall not exceed 0.001.

As approximation for assigning the allowable relative error of the result, we may substitute

$\dfrac{1.3 \times \sqrt{\frac{7}{5}}}{0.4} < \dfrac{1.3 \times 1.2}{0.4} < 4$; that is, 4 is a superior limit of the final result. Hence, the allowable relative error of the result may be taken at $\frac{1}{4000}$. There are 5 single factors to be taken approximately, and there are 5 operations to perform on them after their values are found; we therefore divide the allowable relative error of the result by 10, which gives the limit of the relative error for each factor, $\frac{1}{40000}$. By Rule 4 we may then assume the allowable absolute error of $\sqrt[3]{2}$ at $\frac{1}{40000} = 0.000025$, that of $\sqrt[3]{10}$ at $\frac{2}{40000} = 0.00005$, and that of π at $\frac{3}{40000} = 0.000075$.

Taking then $\sqrt[3]{10}$ with four decimals, $= 2.1544+$ and multiplying by $\pi = 3.1416-$, we have

$$\begin{array}{rll}
2.1544 & + & \\
\underline{\mathit{1}-06141.3} & & \\
646320 & + & \mathit{15} \\
21544 & + & \\
8618 & - & \\
215 & + & \\
\underline{129} & - & \mathit{3} \\
6.76826 & \pm & \mathit{16}
\end{array}$$

Dividing this result by $5.27963+$

```
6.76826 ± 16  |5.27963 +
 527963 +     |1.28196 ± 4
 148863
 105593 −
  43270
  42237 −
   1033
    528 −
    505
    475 −
     30 ± 18
```

Extracting the square root of the quotient,

```
      1.28196±4 (1.13224±3
      1
21)    28
       21
223)   719
       669
2262)  5060
       4524
        536
        452
         84±41
```

Taking $\sqrt[3]{2}$ with five decimals $= 1.25992+$, and multiplying,

```
   1.13224   ±  3
+   29952.1
   1.132240  ± 30
     226448  ±  6
      56612  ±  2
      10190  ±
       1019  ±
         23  ±  5
   1.426532  ± 44
```

Finally, dividing this result by 0.4318965+,

$$
\begin{array}{r|l}
1.426532 \pm 44 & 0.431896{,}5+ \\
\underline{1.295690} & \overline{3.3029 \pm 2} \\
130842 & \\
\underline{129569} & \\
1273 & \\
\underline{864} & \\
409 & \\
\underline{389} & \\
20 \pm 46 &
\end{array}
$$

We may then take for the final result, with three exact decimals, 3.303±, the absolute error being then less than 0.0003, or less than 0.001, the assigned limit.

The reason for Rule 24 may be given in a few words. If a given monomial were composed of any number of approximate factors, either for multiplication or division, and if there were to be no new error made by abridged processes of computation, we might take, for the allowable limit to the relative error of each factor, the allowable relative error of the final result divided by the number of such factors. But since the abridged processes are liable to produce further errors, we must allow for them in assigning the limit of error of the factors, which Rule 24 evidently does. But since, as the operation proceeds, the error increases, it would not even with this allowance be safe to let the new error, produced by each abridged process, be equal to that due to the error of the two factors on which we are operating. How much less than this the error of each operation ought to be made will depend on the degree of complexity of the expression which we have to compute. For all ordinary cases, if nine-tenths of the error of each partial result is due to the errors of the two factors which

give it, there will evidently be a sufficient margin of safety.

If in the statement of a problem the limit of *relative* error of the result be assigned in advance, we may evidently employ the method of Rule 24, except that we are saved the necessity of making the rough calculation of a limit of the result to begin with.

EXAMPLE 65. Compute

$$\frac{\sqrt{857}\times\sqrt[3]{9847.27}}{\pi},$$

with a relative error in the result less than $\frac{1}{10000}$.

There are 3 factors, and 2 operations after they are found, hence the allowable absolute error of $\sqrt{857}$ may be taken at $\frac{29}{50000} = 0.00058$, that of $\sqrt[3]{9847.27}$ at $\frac{20}{50000} = 0.0004$, and that of π at $\frac{3}{50000} = 0.00006$.

Take, therefore, $\sqrt{857}$ with 3 exact decimals, $\sqrt[3]{9847.27}$ with 4, and π also with 4; make the multiplication so as to have 3 decimals in the product, and the division so as to have 2 decimals in the quotient, and the result will be $199.73 \pm \mathit{1}$.

EXAMPLE 66. Compute within 0.0001 the radius of a circle whose area shall be equal to that of a regular hexagon, one side of which equals 1.

If R be the radius, we may easily obtain the formula, $R = \sqrt{\frac{3\sqrt{3}}{2\pi}}$.

We see that the result will be less than 1, hence the allowable relative error of the result is $\frac{1}{10000}$. And, since we may regard the relative error of a sq. root as equal to half that of the number, this expression not being very complex we may let $\frac{1}{10000}$ stand as the allowable relative error of the quantity under the large radical. Further, since multiplying and dividing by

exact factors does not alter the relative error, we need to allow for the error of only one operation on the two approximate factors, before taking the final square root. We have then for the allowable relative error of each of these factors $\frac{1}{30000}$. It is just as well to write $R = \sqrt{\frac{\sqrt{27}}{2\pi}}$, and we may then make the absolute error of $\sqrt{27}$ equal to $\frac{5}{30000}$, say 0.0001, and that of 2π equal to $\frac{6}{30000}$, say 0.0002. Take, therefore, 4 decimals each in $\sqrt{27}$ and 2π, make the division so as to retain 4 exact decimals in the quotient, and extract the root so that the error of it shall be mostly due to that of the quantity. (Compare Art. 41). The answer, with 4 exact decimals, is 0.9093+.

EXAMPLE 67. Compute within 0.001 the radius of a sphere, whose volume shall be equal to that of a truncated pyramid, the altitude of which is 0.752, the bases being regular hexagons, whose sides are respectively 1.42 and 0.843.

The formula will be

$$R = \frac{\sqrt{3}}{2} \times \sqrt[3]{0.752\,\frac{(1.42)^2+(1.42)(0.843)+(0.843)^2}{\pi}}$$

It is not difficult to see that the result will be less than 1. Then the allowable relative error of the result will be $\frac{1}{1000}$. There are 3 factors that will be taken approximately, viz., $\sqrt{3}$, π, and $[(1.42)^2+(1.42)(0.843)+(0.843)^2]$, and we allow for 3 operations, besides those with the exact factors, so that the relative error of each of the approximate factors may be made $\frac{1}{6000}$. The factor just written in brackets will be greater than 3, hence its allowable absolute error will be $\frac{3}{6000} = 0.0005$. The multiplications and additions to obtain this factor may then as well be made exactly, but only 3 exact decimals need

be retained in the value of it when found. The answer within 0.001 is $R=0.839+$.

EXAMPLE 68. Find within 0.0001 the radius of a sphere inscribed in a cone whose altitude is equal to the diameter of its base, and whose volume is equal to $\sqrt[3]{9}$

The formula will be

$$R = \sqrt[3]{\frac{\sqrt[3]{9}}{2\pi\sqrt{3}}}$$

EXAMPLE 69. The volume of a gas at a temperature $t°$, and pressure H^{mm}, being V', the volume at temperature $0°$, and pressure 760^{mm} is given by the formula.

$$V = \frac{V'H}{760(1+t\times 0\cdot 0036650)}$$

Supposing $V' = 1056.7....$, $H = 202.8...$ and $t = 150.1...$, determine how many more decimals would have to be given in each of these quantities, so that V could be calculated with an absolute error not exceeding 0·002. (Proceed by finding the allowable limits of error of the factors, that of the result being 0.002.)

EXAMPLE 70. Compute within one cent the amount of eighty-nine dollars and thirty-seven cents at compound interest for three years and three hundred and forty-seven days, at the rate of seven and one-half per cent. per annum.

Denoting the amount by A, we shall have

$$A = 89\cdot 37\,(1\cdot 075)^3\left(1+\frac{0.075\times 347}{365}\right).$$

53. If a polynomial be proposed, whose terms may be found as accurately as we please, and if it is to be calculated with a relative error in the result not exceeding an assigned limit, then, before we can

assign the allowable limit to the error of each term, we have to determine a rough inferior limit of the final result, from which the allowable absolute error of this result may be found; and from this, as in addition and subtraction, the allowable absolute error of each term, and then, if necessary, the allowable relative error of each term, and so that of each factor in each term.

EXAMPLE 71. Compute the length of one side of a regular pentedecagon inscribed in a circle whose radius is 1, with a relative error in the result less than $\frac{1}{1000}$.

The formula will be

$$S = \frac{1}{4}\sqrt{10+2\sqrt{5}} - \frac{1}{4}\sqrt{3}\,(\sqrt{5}-1).$$

We can see by a little trial that the first of these two terms will be greater than 0.8, and that the second will be less than 0.6; hence, the result will be greater than 0.2, which is, therefore, an inferior limit of the result. We may then, by rule 4, assume the allowable absolute error of the result at $\frac{0.2}{1000} = 0.0002$. The absolute error of each of the two terms may then be 0.0001. The allowable errors of the approximate factors may now be assigned, and the result computed.

EXAMPLE 72. Compute the expression:

$$\frac{\pi}{\sqrt{2}}(\sqrt[3]{2} + \sqrt[3]{10}) + \frac{\pi^2 + \sqrt{2}}{\sqrt{7} - \sqrt{6}} + \sqrt[3]{32} - \sqrt{10},$$

with a relative error in the result less than $\frac{1}{100000}$.

In solving such a problem as this, it is well to arrange the work on paper in such a way that after having carried the computation of each term far enough to serve for assigning an inferior limit of the result, from which to determine the allowable errors,

the computations may be resumed again at the same points, without having to repeat any of the work.

54. If it be required to compute as accurately as possible the value of a complex expression which contains factors whose values are only known with a limited degree of approximation, it is sometimes convenient to determine in advance the degree of precision which we may expect in the result; while, in other cases, the form of the expression is such that the most convenient way is to proceed directly with the computation, taking care that the errors introduced by abridged processes are made small in comparison with those due to the errors of the quantities, and indicating the possible error of each partial result when found, and lastly that of the final result. If a monomial contains a factor whose relative error is very much greater than those of the other factors, it will generally be useless to retain in these latter all the figures that might be taken, although to obtain the result as closely as may be it is best to work with one or two redundant figures. Or, if, in a series of terms for addition or subtraction, one of them would have a much greater absolute error than the others, then it will, of course, avail little to compute these latter to a much lower order of units than can be found in the term having the greatest absolute error.

Example 73. An iron cylinder, weighing 6 kilogrammes, is found to be lengthened 2 millimetres by passing from the temperature of melting ice to that of boiling water. Considering the specific gravity of iron, and the coefficient of dilatation, the radius c the cylinder in decimetres may be computed from the formula

$$R = \sqrt{\frac{6}{7.8 \times 16.920 \times \pi}}$$

Assuming that the factors 7.8 and 16.920 are liable to an absolute error of half a unit of the lowest order in each, determine what degree of relative and absolute approximation may be expected in the value of R, and then make the computation.

The error of the result will evidently be almost entirely due to that of the factor 7.8. If the computation were to be made with the greatest possible precision, we might then expect the limit of relative error of the result to be about $\frac{1}{2} \cdot \frac{5}{750} = \frac{1}{300}$. We can see that the quantity under the radical will not differ much from $\frac{1}{64}$, hence the result will be about 0.12, and the absolute error, therefore, approximately $\frac{0.12}{300} = 0.0004$. We can probably employ the abridged modes of computation without increasing the error to more than 0.0005, which will enable us to get the result within half a unit of thousandths place.

Taking $\pi = 3.142-$, the denominator may be found $= 414.6 \pm \mathit{30}$, the quotient of 6 by this $= 0.01447 \pm \mathit{11}$ and the square root of this $= 0.1203 \pm \mathit{5}$.

EXAMPLE 74. Determine the limit of relative and absolute error of the value of the following expression, supposing the computations were to be exactly made; then make the computations by abridged processes, having regard to Rules 11 and 16, and determine by inspection or otherwise the limit of error of the result as obtained.

$$\frac{(67.348 \pm \mathit{1})\,(6.3261 \pm \mathit{1})\,(1.7253 \pm \mathit{1})}{(4.2785 \pm \mathit{1})\,(0.62734 \pm \mathit{1})}.$$

Taking the sum of limits of the relative errors of all the factors, we have $\frac{1}{60000} + \frac{1}{60000} + \frac{1}{15000} + \frac{1}{40000} + \frac{1}{60000} = \frac{17}{120000}$. But $\frac{17}{120000} < \frac{1}{7000}$; hence the relative error of the result, if no considerable new error is introduced in the computation, will not

exceed $\frac{1}{7000}$; and since the result will be less than 280, the absolute error need not exceed $\frac{280}{7000} = 0.04$.

EXAMPLE 75. If L and B denote the length and breadth of one base of a prismoid, l and b the length and breadth of the opposite base, and h the distance between the bases, the volume of the prismoid is given by the following formula:

$$V = \frac{1}{6}h\,(2BL+2bl+Bl+bL).$$

Assume $h=10.20\pm 1$, $B=7.84\pm$, $L=9.92\pm$, $b=6.43\pm 2$, and $l=8.495\pm$; compute the value of V with what accuracy the data allow, and exhibit the possible error.

EXAMPLE 76. The weight P of a volume of air V, saturated with aqueous vapor, at temperature t and pressure H, the tension of the vapor being F, and the coefficient of expansion of air being a, is given by the formula

$$P = \frac{0.31\,V}{(1+at)760}\,(H-\tfrac{3}{8}F).$$

Let $V=1000$, $F=773.71$, $H=1500$, $a=0.003665$, and $t=100.5$, these being supposed liable to error of half a unit of the lowest order in each, but the other numbers in the formula being supposed exact. Find the value of P, and its limit of error.

Ans. 360.7 ± 5.

EXAMPLE 77. The formula for computing heights by the barometer, in feet, is

$$x = 60346(1+0.002560\cos 2\phi)\left(1+\frac{2\,(t+t')}{1000}\right)\log\frac{h\left(1-\dfrac{t}{6500}\right)}{h'\left(1-\dfrac{t'}{6500}\right)}$$

Let $h=30.025$, $h'=28.230$, $t=17.32$, and $t'=10.55$, the limit of error of these quantities being 5 units

of the lowest order in each, as also of the factors 60346 and 0.002560. Find the value of x when $\phi = 65°$, and show the limit of error of the result by observing the possible error at each step of the work.

Ans. 1674±*11*.

EXAMPLE 78. The interior diameter of a copper ring being 18 centimetres at the temperature of 10°, and the diameter of an iron sphere being 18.05 at the same temperature, to what common temperature must they be heated before the latter will pass through the former?

From a consideration of the coefficients of expansion of copper and iron, the following formula is given:

$$x = \frac{18.05 \times 1.000170 - 18 \times 1.000126}{18 \times 1.000126 \times 0.0000170 - 18.05 \times 1.000170 \times 0.0000126}.$$

Required to compute the value of x, under the supposition that all the factors except 18 and 18.05 are liable to error of 1 of their lowest units, and to exhibit the limit of error of the result.

Ans. 646°±*35*°.

EXAMPLE 79. The density D_4, of the air in a certain air-pump, after four strokes of the piston, is given by the formula

$$D_4 = \left(\frac{3-0.885}{3}\right)^4 - \frac{0.005}{0.885}\left[1 - \left(\frac{3-0.885}{3}\right)^4\right]$$

Compute D_4, supposing that where the quantity 0.885 occurs in the formula it is liable to an error equal to 0.0005, and exhibit the limit of error of the result.

EXAMPLE 80. The index of refraction of a prism being given by the formula

$$n = \frac{\sin \frac{1}{2}(\phi + a)}{\sin \frac{1}{2}\phi},$$

5

suppose $\phi=60°\ 00'\ 46''\pm 30''$, $\alpha=47°\ 41'\ 50''\pm 30''$, compute n by natural or logarithmic functions, and exhibit the limit of error.

EXAMPLE 81. Having given the three sides of a spherical triangle, $a=74°\ 23'$, $b=35°\ 46'\ 14''$, and $c=100°\ 39'\ 25''$, the formula for determining the angle A, is

$$\cos\tfrac{1}{2}A=\sqrt{\frac{\sin\frac{1}{2}s\sin\left(\frac{1}{2}s-a\right)}{\sin b\sin c}},$$

in which $s=a+b+c$. Supposing the given sides to be each liable to an error of 30″, compute the angles A, B, and C by natural and by logarithmic functions, and exhibit their limits of error.